Overview of the course:

From the Brunn-Minkowski inequality to elliptic PDE's

Geometry and Analysis Seminar
Tohoku University - 9-11 March 2015

The course will focus on some of the numerous and deep relations existing between convex geometry (and in particular the Brunn-Minkowski inequality), and some parts of the theory of elliptic PDE's and calculus of variations. A possible plan of the lectures follows[1].

Lecture 1 (9/3/2015). We will start presenting some basic facts of convex geometry. We will introduce the class $\mathcal{K}^n$ of n-dimensional convex bodies (compact convex subsets of $\mathbf{R}^n$), along with the Minkowski addition. We will also define the notion of support function of a convex body, which will be used several times in the course. Through the Steiner formula we will introduce a class of important global geometric quantities connected to a convex body: the *intrinsic volumes*, and describe their main properties. Analogously, from the more general polynomial expansion of the volume of a linear combination of convex bodies we will derive the definition of *mixed volumes*.

Lecture 2 (9/3/2015). This lecture will be entirely devoted to the core of the course: the **Brunn-Minkowski inequality** (in brief: (BM)). This inequality can be rephrased saying that the volume (i.e. the Lebesgue measure, denoted by V_n) to the power $1/n$, is *concave* in $\mathcal{K}^n$, with respect to the Minkowski addition. This is one of the cornerstones of convex geometry and, surprisingly enough, creates a strong bridge toward the theory of elliptic PDE's and calculus of variations, as we will see later in the course. In this lecture we will focus on: (1) the proof of (BM), which relies on its functional form, the **Prékopa-Leindler inequality**; (2) the link with the **isoperimetric inequality**, which can be easily proved using (BM); (3) the **Aleksandrov-Fenchel inequalities**, an important class of inequalities for mixed volumes, which "contain" the (BM) inequality.

Lectures 3 and 4 (10/3/2015). This lectures will mark the passage from convex geometry to elliptic PDE's and calculus of variations. We will start with the notion of Brunn-Minkowski type inequality for a generic functional $\mathbf{F}$. We will see some example of *geometric* functionals verifying this type of inequality and then we will pass to the three main variational functionals for which a (BM) type inequality holds: the **first eigenvalue of the Laplace operator**, with Dirichlet boundary conditions; the **2-capacity** and the **torsional rigidity**. Time permitting, we will describe the techniques of proof that have been employed to prove (BM) for these functionals. In the case of the eigenvalue of the Laplace operator, we will also see the proof given by Brascamp and Lieb of the log-concavity of the first (positive) eigenfunction in a convex domain. Moreover, we will indicate the extensions of the previous results that have been achieved in the last years. In the final part we will describe some functionals for which a (BM) inequality *does not* hold.

Lecture 5 (11/3/2015). The subject of this lecture will be **Hadamard's type formulas**. A Hadamard's formula is typically an integral representation for the derivative of a functional

[1]This program is quite ambitious and maybe not all the topics will be covered.

F, with respect to perturbations of the domain. In some cases these formulas permits to define the *first variation* of **F**. A well-known formula of this kind in convex geometry is the following one, involving the volume:

$$\frac{d}{d\epsilon}V_n(K+\epsilon L)|_{\epsilon=0^+} = \int_{\mathbf{S}^{n-1}} h_L(\xi)\, d\sigma_k(\xi),$$

here K and L are convex bodies, h_L is the support function of L, and σ_K is the so-called *area measure* of K. According to the previous formula, σ_k is interpreted at the first variation of the volume, computed at K. In this lecture we will see various Hadamard's formulas, also concerning the variational functionals that we have encountered in the previous lectures. This will give us the occasion to speak about the **Minkowski problem**, which consists in finding a convex body K, assigned the first variation of the volume at K, i.e. its area measure σ_K, and more generally about Minkowski type problems for variational functionals.

Lecture 6 (11/3/2015). This lecture will be devoted to explore the **infinitesimal form** of the Brunn-Minkowski inequality, and to see that this is equivalent to a class of **inequalities of Poincaré type**. As we said, (BM) states that the the volume V_n to the power $1/n$ is concave in $\mathcal{K}^n$. If we restrict ourselves to a suitable subclass of convex bodies (namely, those with C^2 boundary and strictly positive Gauss curvature), and we identify each convex body with its support function, we can compute the *second variation* of $V_n^{1/n}$, which turns out to be a quadratic form acting on test functions. If we impose that this quadratic form is negative semi-definite (as a consequence of (BM)), we find a family of functional inequalities on the unit sphere $\mathbf{S}^{n-1}$ of $\mathbf{R}^n$, including the standard Poincaré inequality with the right constant.