FINITE GAP SOLUTIONS FOR HORIZONTAL MINIMAL SURFACES OF FINITE TYPE IN 5-SPHERE

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Abstract. We consider a horizontal minimal surface in 5-sphere $s_0 : M \longrightarrow S^5 \subset \mathbf{C}^3$. The Gauss, Codazzi and Ricci equations for s_0 becomes a unified equation called "Tzitéica equation" of elliptic type as follows.

$$\partial_{\overline{z}}\partial_z \mathfrak{u} = e^{-2\mathfrak{u}} - e^{\mathfrak{u}}, \tag{1}$$

where $z = x + \sqrt{-1}y$ a local coordinate system for a Riemann surface M and $\mathfrak{u} = \mathfrak{u}(z, \overline{z})$ is a real valued function on M. The induced metric g on M is given by $g = 2e^{\mathfrak{u}}dzd\overline{z}$. Let $< , >$ be a Hermitian fibre metric on $M \times \mathbf{C}^3$ compatible with g. Since s_0 is a minimal isometric immersion, we have

$$< s_0, s_0 >= 1, \quad \partial_{\overline{z}}\partial_z s_0 = -e^u s_0. \tag{2}$$

The horizontality of s_0 with respect to the Hopf fibration $S^5 \longrightarrow \mathbf{C}P^2$ means that

$$< \partial_z s_0, s_0 >= 0, \quad < \partial_{\overline{z}} s_0, s_0 >= 0. \tag{3}$$

We also have $< s_0, \partial_{\overline{z}} s_0 >= 0, < s_0, \partial_z s_0 >= 0$. The conformality of s_0 means that

$$< \partial_z s_0, \partial_{\overline{z}} s_0 >= 0, \quad < \partial_{\overline{z}} s_0, \partial_z s_0 >= 0. \tag{4}$$

We set

$$s_1 = e^{-\frac{\mathfrak{u}}{2}}\partial_z s_0, \quad s_2 = e^{-\frac{\mathfrak{u}}{2}}\partial_{\overline{z}} s_0, \quad \phi = e^{\frac{\mathfrak{u}}{2}} < \partial_z\partial_z s_0, s_2 > . \tag{5}$$

It then follows from (2), (3), (4) and (5) that $F = (s_0\ s_1\ s_2)$ is a unitary frame on $M \times \mathbf{C}^3$. We have

$$F^{-1}\partial_z F = \begin{pmatrix} 0 & 0 & -e^{\frac{\mathfrak{u}}{2}} \\ e^{\frac{\mathfrak{u}}{2}} & \frac{\mathfrak{u}_z}{2} & 0 \\ 0 & \phi e^{-\mathfrak{u}} & -\frac{\mathfrak{u}_z}{2} \end{pmatrix}, \quad F^{-1}\partial_{\overline{z}} F = \begin{pmatrix} 0 & -e^{\frac{\mathfrak{u}}{2}} & 0 \\ 0 & -\frac{\mathfrak{u}_{\overline{z}}}{2} & -\overline{\phi} e^{-\mathfrak{u}} \\ e^{\frac{\mathfrak{u}}{2}} & 0 & \frac{\mathfrak{u}_{\overline{z}}}{2} \end{pmatrix}, \tag{6}$$

where we have set $\mathfrak{u}_z = \partial_z \mathfrak{u}, \mathfrak{u}_{\overline{z}} = \partial_{\overline{z}} \mathfrak{u}$. If we set $U = F^{-1}\partial_z F, V = F^{-1}\partial_{\overline{z}} F$, then the compatibility condition for (6) is given by $\partial_{\overline{z}} U - \partial_z V = [U, V]$, which is equivalent to the following equations.

$$\begin{cases} \partial_{\overline{z}}\partial_z \mathfrak{u} & = |\phi|^2 e^{-2\mathfrak{u}} - e^{\mathfrak{u}}, \\ \partial_{\overline{z}}\phi & = 0. \end{cases}$$

Changing the local complex coordinate apropriately, we may assume that $\phi = -1$. We then obtain the Tzitzéica equation stated in (1). This is a special case of the

famous Toda equation in the theory of the integrable systems. We first give an explicit solution of the Tzitzéica equation in terms of the Jacobi elliptic function. Secondly, we express the solution in terms of the Riemann theta function, which is so-called a finite gap solution. Moreover, some examples of horizontal minimal surfaces in 5-sphere can be described in terms of the Jacobi elliptic functions, which are also described explicitly in terms of the Baker-Akhiezer function. In this work, we give a spectal curve explicitly, which is a hyperelliptic curve of genus 2 and given in the affine coordinate by $\hat{\mathcal{C}} : \widetilde{\nu}^2 = \prod_{j=1}^{3} (\mu - \mu_j)(\mu + \mu_j)$. We also give Abelian differentials of second kind explicitly. Our Baker-Akhiezer function is given by

$$\hat{\Psi}(z, \overline{z}, \hat{P}, \mathbf{e}) = \frac{\theta(\mathcal{B}(\hat{P}) - (z + \overline{z})\mathbf{U}^0 - \mathbf{e})\,\theta(\mathbf{e})}{\theta(\mathcal{B}(\hat{P}) - \mathbf{e})\,\theta((z + \overline{z})\mathbf{U}^0 + \mathbf{e})}\,\Phi_e(z, \overline{z}, \hat{P}),$$

where $\mathbf{e} = \pi\sqrt{-1}, \mathbf{U}^0 = \dfrac{\pi\sqrt{-1}}{2\omega^0}$, $\hat{P} \in \hat{\mathcal{C}}$ and $\mathcal{B} : \hat{\mathcal{C}} \longrightarrow \mathrm{Prym}(\hat{\mathcal{C}})$ is the Prym-Abel map. $\Phi_e(z, \overline{z}, \hat{P})$ is given by

$$\Phi_e(z, \overline{z}, \hat{P}) = \exp\left(z \left(\int_{\hat{P}_1}^{\hat{P}} \hat{\Omega}_\infty - \frac{\sqrt{-1}}{2}\mu_1 \right) - \overline{z} \left(\int_{\hat{P}_1}^{\hat{P}} \hat{\Omega}_0 - \frac{\sqrt{-1}}{2}\mu_1 \right) \right).$$

Finally, remark that a generalization of the reconstruction from some spectral curve of higher genus is also possible.

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