Orthogonality of subalgebras and Latin squares

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September, 1991

By a $*$-subalgebra of the matrix algebra $M_n(\mathbf{C})$ we mean a subalgebra containing the identity closed under $*$ (conjugate transpose). A maximal abelian $*$-subalgebra of $M_n(\mathbf{C})$ is unitarily conjugate to the subalgebra of diagonal matrices. Two maximal abelian $*$-subalgebras A, B of $M_n(\mathbf{C})$ are said to be *orthogonal* (or an *orthogonal pair*) if the following condition is satisfied.

$$\mathrm{tr}(x)\mathrm{tr}(y) = n\mathrm{tr}(xy) \quad \text{for all } x \in A,\ y \in B.$$

Geometric meaning of the above condition is that $A \cap 1^{\perp}$ and $B \cap 1^{\perp}$ are orthogonal with respect to the inner product $\langle x|y\rangle = \mathrm{tr}(xy^*)$. For example, the subalgebra A of diagonal matrices is orthogonal to the subalgebra B generated by the matrix

$$\begin{pmatrix} & & & 1 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{pmatrix}$$

A pair of orthogonal maximal abelian $*$-subalgebras will be called *standard* if it is unitarily conjugate to the above pair (A, B). In 1983 Popa conjectured that if n is a prime, then any pair of orthogonal maximal abelian $*$-subalgebras of $M_n(\mathbf{C})$ is standard. The conjecture have been disproved, however.

Theorem 1 (de la Harpe–Jones [1], Munemasa–Watatani [4]) *If n is a prime and $n \geq 7$ then there exists a non-standard orthogonal pair of maximal abelian $*$-subalgebras of $M_n(\mathbf{C})$.*

The construction of non-standard pair in the above theorem is based on Paley graph on n vertices.

Recently Kawanaka pointed out that our work is closely related to the work by Kostrikin et al [3], which we shall discuss below. An orthogonal decomposition of $\mathfrak{sl}(n, \mathbf{C})$ is a decomposition of $\mathfrak{sl}(n, \mathbf{C})$ into the direct sum of mutually orthogonal Cartan subalgebras. Note that $\mathfrak{sl}(n, \mathbf{C})$ is equipped with the Killing form with respect to which orthogonality of Cartan subalgebra is defined. An orthogonal decomposition of $\mathfrak{sl}(n, \mathbf{C})$ exists if n is a prime power, but the existence is unknown if otherwise.

Popa [5] proposes an equivalent problem independently, in terms of maximal abelian $*$-subalgebras, i.e., Popa asks the maximum number of mutually orthogonal maximal abelian $*$-subalgebras of $M_n(\mathbf{C})$. This problem can be stated in terms of Cartan subalgebras of $\mathfrak{sl}(n, \mathbf{C})$ because of the following proposition.

Proposition 2 *If H_1 and H_2 are orthogonal Cartan subalgebras of $\mathfrak{sl}(n, \mathbf{C})$, then $\langle H_1, 1\rangle$ and $\langle H_2, 1\rangle$ are orthogonal pair of maximal abelian $*$-subalgebras of $M_n(\mathbf{C})$. Conversely, if A, B are mutually orthogonal maximal abelian $*$-subalgebras of $M_n(\mathbf{C})$, then $A \cap \mathfrak{sl}(n, \mathbf{C})$ and $B \cap \mathfrak{sl}(n, \mathbf{C})$ are orthogonal Cartan subalgebras of $\mathfrak{sl}(n, \mathbf{C})$.*

Proposition 3 *There exists an orthogonal decomposition of $\mathfrak{sl}(n, \mathbf{C})$ if and only if there exist $n + 1$ pairwise orthogonal maximal abelian $*$-subalgebras of $M_n(\mathbf{C})$.*

There is a remarkable similarity between the existence of an orthogonal decomposition of $\mathfrak{sl}(n, \mathbf{C})$ and the existence of a complete set of mutually orthogonal Latin squares. Compare Proposition 2, Proposition 3 with Proposition 4, Proposition 5.

Consider $\mathbf{C}^{n^2}$ as a commutative $\mathbf{C}$-algebra and equip it with the natural hermitian inner product. Two subalgebras A, B are said to be orthogonal if $A \cap 1^\perp$ and $B \cap 1^\perp$ are orthogonal. With this convention we have the following.

Proposition 4 *There exist s mutually orthogonal Latin squares of order n if and only if there exist $s + 2$ pairwise orthogonal n-dimensional subalgebras of $\mathbf{C}^{n^2}$.*

Proposition 5 *There exists a complete set of mutually orthogonal Latin squares of order n if and only there exist $n+1$ pairwise orthogonal n-dimensional subalgebras of $\mathbf{C}^{n^2}$.*

In 1987 D. N. Ivanov showed that an orthogonal decomposition of $\mathfrak{sl}(n, \mathbf{C})$ can be constructed from certain translation planes, and from non desargusian planes one can obtain a "non desarguesian" orthogonal decomposition of $\mathfrak{sl}(n, \mathbf{C})$, i.e., not unitarily conjugate to the one obtained from the desargusian plane.

Open Questions (i) From an affine plane (not necessarily a translation plane) can one construct orthogonal decomposition of $\mathfrak{sl}(n, \mathbf{C})$?

(ii) Does there exist an orthogonal decomposition of $\mathfrak{sl}(n, \mathbf{C})$ if n is not a prime power?

References

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