Flag-Transitive 2-Designs

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Flag-transitive 2-$(v, k, 1)$ design

$v = 2^{2n}$, $k = 4$, $\lambda = 1$

point-set $= GF(2^{2n})$

line-set is a set of affine 2-dim $GF(2)$-subspaces

Example $AG(n, 4)$

Main Result

Enumeration of all such designs $\mathcal{D}$ with

$$AG^3L(1, v) \subset \mathrm{Aut}\,\mathcal{D} \subset A\Gamma L(1, v)$$

Buekenhout - Delandsheer - Doyen - Kleidman - Liebeck - Saxl (1990)

announced

"All flag-transitive 2-$(v, k, 1)$ designs are known except those $\mathcal{D}$ with

$$\mathrm{Aut}\,\mathcal{D} \subset A\Gamma L(1, v)$$

$$K = GF(v)$$

$$A\Gamma L(1, v) = \left\{ x \mapsto a x^{\psi} + b \;\middle|\; \begin{array}{l} a, b \in K,\ a \neq 0 \\ \psi \in \mathrm{Aut}\,K \end{array} \right\}$$

$$A G^3 L(1, v) = \left\{ x \mapsto a^3 x + b \;\middle|\; a, b \in K,\ a \neq 0 \right\}$$

$$v - 1 \equiv 0 \pmod 3$$

$|AG^3L(1, v)|$ = the number of flags of a 2-$(v, 4, 1)$ design

$$= v(v-1)/3$$

A line-spread in $PG(n-1, q)$ is a collection of lines which partition the set of points.

Example in $PG(2n-1, q)$

1-dim. $GF(q^2)$-subspaces of $GF(q^{2n})$

$\Downarrow$

line spread which forms an orbit under the Singer cycle.

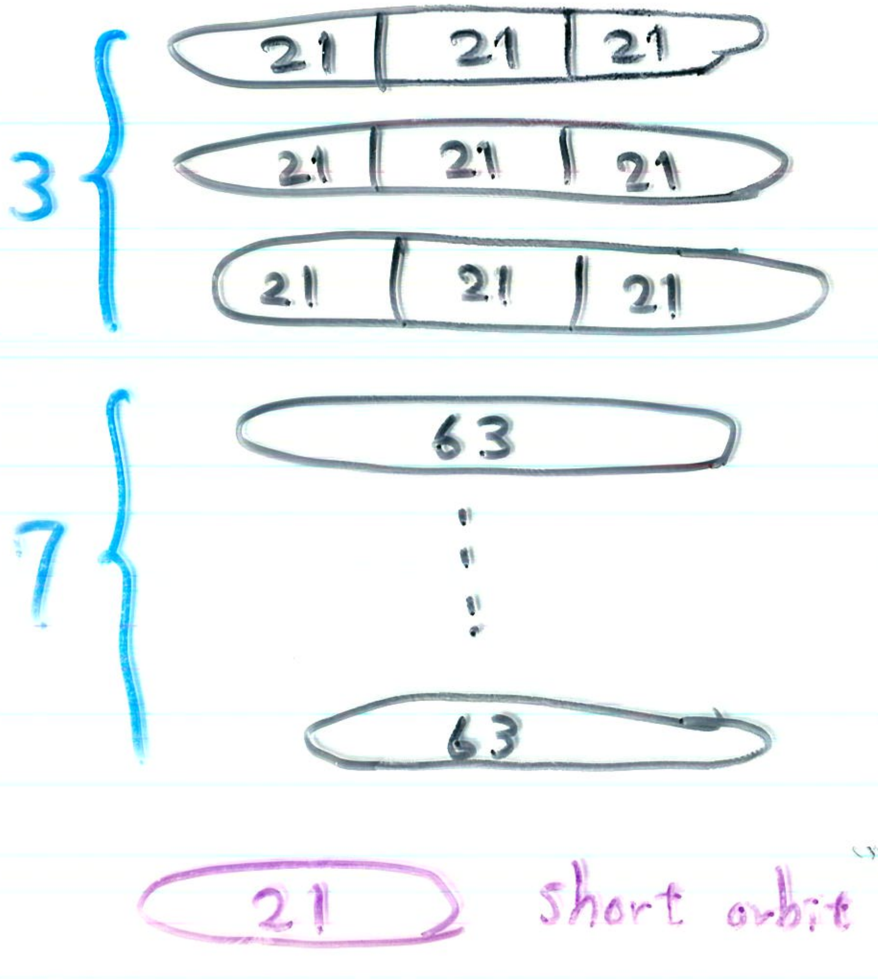
This is the only line-spread invariant under the Singer cycle.

A parallelism is a partition of the set of lines into line-spreads.

Existence of parallelism in PG(2n-1, q)

[Baker, Beutelspacher, ...
Prince, Penttila-Williams
open in general]

Sarmiento found new point-cyclic parallelism in PG(5, 2)



651 lines

⇓

11 orbits under Singer cycle

$PG(2n-1,2)$ has $2^{2n}-1$ points

a line has 3 points

a line-spread has $\dfrac{2^{2n}-1}{3}$ lines.

Problem Find all line-spreads whose members are permuted cyclically by the subgroup of index 3 of the Singer cycle.

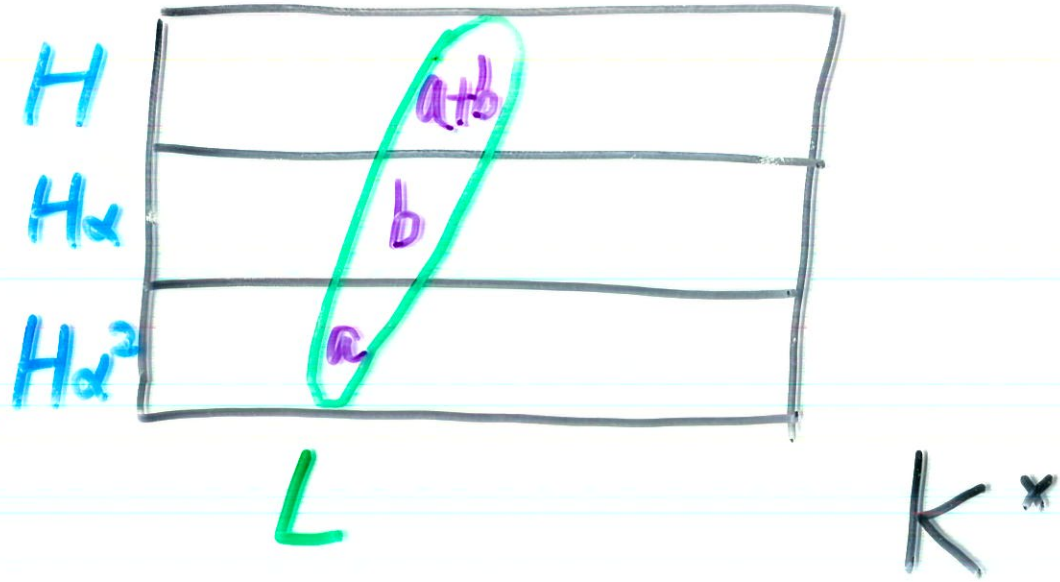
$\Updownarrow$

order $\dfrac{2^{2n}-1}{3}$

Assume $q = 2$ $\quad K = GF(2^{2n})$ ⑧

$$PG(2n-1, 2) = K^{\times} \supset H$$

$$K^{\times} = \langle \alpha \rangle, \quad H = \langle \alpha^3 \rangle$$



L^H is a line-spread

$\iff$ L is a transversal to $K^{\times}/H$

The number of

$= |\{(a, b) \in H\alpha \times H\alpha^2 \mid a+b \in H\}|$

(cyclotomic number)

Theorem In $PG(2n-1,2)$ the number of line-spreads whose members are permuted cyclically by the subgroup of index 3 of the Singer cycle is $\dfrac{(2^n+(-1)^{n+1})^2}{9}$

Example

$n=2$ $PG(3,2)$ $\dfrac{(4-1)^2}{9}=1$

only trivial example

$n>2$ $\dfrac{(2^n+(-1)^{n+1})^2}{9}>1$

always nontrivial example

$$PG(2n-1,2) \longleftrightarrow K^{\times}$$

$$\cap$$

$$K = GF(2^{2n})$$

If L is a line belonging to a line-spread satisfying the condition of **Theorem** then $(K, \mathcal{B}(L))$ is a flag-transitive 2-$(2^{2n}, 4, 1)$ design, where

$$\mathcal{B}(L) = \{ h(L \cup \{0\}) + k \mid h \in H,\ k \in K \}$$

$$K \rtimes H = A\Gamma L(1, 2^{2n})$$

Prop. $(K, \mathcal{B}(L)) \cong AG(n, 4)$

$\Longleftrightarrow$ $n \not\equiv 0 \pmod 3$ and

$L \in K^\times / GF(4)^\times$

Almost all (if $n \equiv 0 \pmod 3$ then actually all) of $(2^n + (-1)^{n+1})^2 / 9$ line-spreads define a flag-transitive 2-$(2^{2n}, 4, 1)$ design not isomorphic to $AG(n, 4)$

$B(L) = B(L')$

$\Longleftrightarrow$ L, L' belong to the same H-orbit.

$(X, B(L)) \cong (X, B(L'))$

$\Longleftrightarrow$ $\exists\, j, k \in \mathbb{Z}$

Frobenius automorphism

$K^\times \ni \beta^j (L^{2^k}) = L'$

$\Longleftrightarrow$ L, L' belong to the same $(K^\times \cdot \mathrm{Aut}\,K)$-orbit.

$\Gamma L(1, 2^{2n})$

$|K^\times \cdot \mathrm{Aut}\,K : H| = 6n$

Theorem The number of pairwise nonisomorphic flag-transitive 2-$(2^{2n}, 4, 1)$ designs whose automorphism group is a subgroup of $A\Gamma L(1, 2^{2n})$ is at least

$$\begin{cases} \left\lceil \dfrac{(2^n + (-1)^{n+1})^2}{54n} \right\rceil & \text{if } n \equiv 0 \pmod{3} \\[2ex] \left\lceil \dfrac{(2^n + (-1)^{n+1})^2 - 9}{54n} \right\rceil & \text{if } n \not\equiv 0 \pmod{3} \end{cases}$$

Jumela Sarmiento (1999)

- found a recursive formula for the number of designs
- gave an explicit construction for the designs $\mathcal{D}$ with

$$|A\Gamma L(1,2^{2m}) : \mathrm{Aut}\,\mathcal{D}| = \text{odd}$$

- showed that $\mathcal{D}$ is obtained by Kantor's inflation trick when

$$|A\Gamma L(1,2^{2m}) : \mathrm{Aut}\,\mathcal{D}| = \text{even} < 6$$

Note

$$|A\Gamma L(1,2^{2m}) : AG^3L(1,2^{2m})| = 6$$

Open Problems

- Explicit construction of $\mathcal{D}$ with $\mathrm{Aut}\,\mathcal{D} = A\Gamma^3L(1, 2^{2m})$

- Complete classification of flag-transitive 2-$(v, 4, 1)$ designs
  - v = power of 2, $\mathrm{Aut}\,\mathcal{D} \not\supseteq A\Gamma^3L(1, 2$
  - v = power of 2, blocks are not affine subspaces
  - v = odd prime power

- Line-spreads in $PG(2n-1, q)$ whose members are permuted cyclically by the subgroup of index $q+1$ of the Singer cycle.