Flag-Transitive
2-$(v,4,1)$ Designs

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Want to find (or classify if possible) good designs.

$\uparrow$

with large automorphism group

$\uparrow$

doubly transitive on points

$\Downarrow$ if $\lambda = 1$

flag-transitive

$\swarrow \qquad \searrow$

point-transitive $\Leftarrow$ block-transitive

All doubly-transitive 2-$(v, k, 1)$ designs are known. All flag-transitive Steiner triple systems are known.

$k \geq 4$ open

Theorem (Buekenhout, Delandsheer, Doyen, Kleidman, Saxl)

All flag-transitive 2-$(v,k,1)$ designs are known (examples: $PG(n,q)$, $AG(n,q)$ etc.) **EXCEPT**

$v =$ prime power

points $= GF(v)$

$$T \subset \mathrm{Aut}\,\mathcal{D} \subset A\Gamma L(1,v)$$

where

$$T = \{ x \mapsto x + a \mid a \in GF(v) \}$$

$A\Gamma L(1,v)$

$$= \{ x \mapsto a x^{\varphi} + b \mid a, b \in GF(v),\ a \neq 0,\ \varphi \in \mathrm{Aut}\,GF(v) \}$$

Assume

$v = 2^m$, $k = 4$

points $= GF(2^m) = K$

$K^\times = \langle \alpha \rangle$

$\sigma : K \to K$ $\sigma(x) = \alpha x$

so that

$$A\Gamma L(1, v) = T \rtimes \langle \sigma \rangle \rtimes \mathrm{Aut} K$$

Put

$$G = T \rtimes \langle \sigma^3 \rangle$$

note

$|G| =$ the number of flag

Suppose

$\mathcal{D} = (GF(2^m), \mathcal{B})$ is a 2-$(2^m, 4, 1)$ design, on which $G = T \rtimes \langle \sigma^3 \rangle$ acts flag-transitively.

Then for $\forall$ fixed $B \in \mathcal{B}$,

$$\mathcal{B} = \{ g(B) \mid g \in G \}$$

We may assume

$$B \ni 0, 1$$

Then we can show

$$B = \{ 0, 1, \beta, \beta + 1 \}$$

for some $\beta \in GF(2^m)$

(B is an additive subgroup)

Problem

For which $\beta \in GF(2^m)$,

$\mathcal{D} = (GF(2^m), \mathcal{B})$ is a 2-$(v, 4, 1)$ design?

$$\left(\begin{array}{l} \mathcal{B} = \{ g(B) \mid g \in G = T \rtimes \langle \sigma^3 \rangle \} \\ B = \{0, 1, \beta, \beta+1\} \end{array}\right)$$

Answer (not explicit)

$\{1, \beta, \beta+1\}$ is a system of representatives for cyclotomic cosets C_3, that is,

$$GF(2^m) = \langle \alpha^3 \rangle \cup \langle \alpha^3 \rangle \beta \cup \langle \alpha^3 \rangle (\beta + 1)$$

Theorem

The number of $\beta \in GF(2^m)$ s.t. $(GF(2^m), \mathcal{B})$ is a 2-$(2^m, 4, 1)$ design (automatically flag-transitive) is

$$\frac{2}{9}\left(2^n + (-1)^{n+1}\right)^2$$

$m = 2n$

$$\mathcal{B} = \{ g(B) \mid g \in T \rtimes \langle \sigma^3 \rangle \}$$

$$B = \{0, 1, \beta, \beta + 1\}$$

Corollary The number of pairwise nonisomorphic flag-transitive 2-$(2^m, 4, 1)$ designs whose automorphism group satisfies

$$T \rtimes \langle \sigma^3 \rangle \subset \mathrm{Aut}\,\mathcal{D} \subset A\Gamma L(1, 2^m)$$

is at least

$$\left\lceil \frac{(2^m + (-1)^{m+1})^2}{54m} \right\rceil \qquad \text{if } 3 \mid m$$

Note

$L = \{1, \beta, \beta+1\}$ may be regarded as a line in $PG(m-1, 2) = GF(2^m)$

L, L' are in the same $\Gamma L(1, 2^m)$-orbit

$\Longleftrightarrow$

$B = \{0\} \cup L$, $B' = \{0\} \cup L'$ defines isomorphic designs

J. Sarmiento (1999)

- gave a formula for the number of isomorphism classes

- showed that some of the designs are obtained by <u>Kantor's inflation trick</u> applied to the design with smaller m.

Problem Describe all such designs, that is, describe all $\beta \in GF(2^m)$ for which $(GF(2^m), \mathcal{B})$ is a 2-$(2^m, 4, 1)$ flag transitive design, where

$$\mathcal{B} = \{ g(B) \mid g \in T \rtimes \langle \sigma^3 \rangle \}$$
$$B = \{0, 1, \beta, \beta + 1\}$$

Example $m = 6$

$(GF(64), \mathcal{B})$ is a 2-$(64, 4, 1)$ design

$\Longleftrightarrow$ B contains an element (β or $\beta+1$ of order 9

This is a unique flag transitive 2-$(64, 4, 1)$ design not isomorphic to $AG(3, 4)$.