$q = 2^r$, r = positive integer

$u = (u_1, u_2, \ldots, u_{2n}) \in \mathbb{F}_q^{2n}$

$v = (v_1, v_2, \ldots, v_{2n}) \in \mathbb{F}_q^{2n}$

$$(u, v) = \sum_{i=1}^{2n} u_i v_i$$

$$wt(u) = |\{ i \mid u_i \neq 0 \}|$$

$\mathbb{F}_q^{2n} \supset C$ linear code

$$C^\perp = \{ u \in \mathbb{F}_q^{2n} \mid (u, v) = 0 \quad \forall v \in C \}$$

C: self-dual $\Longleftrightarrow$ $C = C^\perp$

For $q = 2$,

C: doubly-even $\Longleftrightarrow$ $wt(u) \equiv 0 \pmod 4$ $\forall u \in C$

Doubly-even self-dual (d.e.s.d) codes are also called Type II codes

Gaborit-Pless-Solé-Atkin (1999)

"Type II" codes over $\mathbb{F}_4$

Mass Formula

d.e.s.d codes of length $2n$ $(q=2)$

$$= \prod_{i=0}^{n-2} (2^i+1)$$

[MacWilliams - Sloane - Thompson (1972), Thompson (1973)]

Type II codes of length $2n$ over $\mathbb{F}_4$

$$= \prod_{i=0}^{n-2} (4^i+1)$$

[Gaborit - Pless - Solé - Atkin (1999)]

$\prod_{i=0}^{n-2} (q^i+1) = \#$ totally singular $(n-1)$-spaces w.r.t. a nondegenerate quadratic form on a $(2n-2)$-dim. space over $\mathbb{F}_q$, of Witt index $n-1$.

[Segre (1959)
Ray-Chaudhuri (1962)
Feng-Dai (1964)]

$\mathbf{1}$ = all one vector $\in \mathbb{F}_2^{2n}$

$$\langle \mathbf{1} \rangle \subset C \subset \langle \mathbf{1} \rangle^{\perp}$$

d.e.s.d dim $C = n$

$$\begin{tikzcd} C \arrow[d, leftrightarrow, "1:1"'] \\ \overline{C} \end{tikzcd}$$

$$\overline{C} \subset \langle \mathbf{1} \rangle^{\perp} / \langle \mathbf{1} \rangle$$

totally singular $(n-1)$-sp
w.r.t. the nondegerate
quadratic form $\overline{f}$

$$\begin{tikzcd} \langle \mathbf{1} \rangle^{\perp} \arrow[r, "f"] \arrow[d] & \mathbb{F}_2 \\ \langle \mathbf{1} \rangle^{\perp} / \langle \mathbf{1} \rangle \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

$f(x) = \frac{wt(x)}{2}$ Broué (1977) Puig?

A basis $B=\{\alpha_1,\alpha_2,\cdots,\alpha_r\}$ of $\mathbb{F}_{2^r}/\mathbb{F}_2$ is called trase-orthogonal if

$$Tr(\alpha_i\alpha_j)=\delta_{ij}$$

$$\mathbb{F}_{2^r} \longrightarrow \mathbb{F}_2^{\,r}$$

$$x=\sum_{i=1}^{r} c_i\alpha_i \longmapsto (c_1,\cdots,c_r)$$

extends to

Gray map $\phi:(\mathbb{F}_{2^r})^{2n}\to\mathbb{F}_2^{\,2nr}$

Definition. A linear code $C\subset(\mathbb{F}_{2^r})^{2n}$ is Type II (w.r.t B) if $\phi(C)$ is d.e.s.d.

$$\langle \mathbb{1} \rangle \subset C \subset \langle \mathbb{1} \rangle^{\perp}$$

$$\begin{array}{c} C \\ \Big\updownarrow 1:1 \\ \bar{C} \subset \langle \mathbb{1} \rangle^{\perp} / \langle \mathbb{1} \rangle \end{array}$$

Type II w.r.t $B = \{\alpha_1, \dots, \alpha_n\}$

dim. $n-1$ dim. $2n-2$

totally singular w.r.t the nondegenerate quadratic form $\bar{f}$

$$\begin{tikzcd} \langle \mathbb{1} \rangle^{\perp} \arrow[r, "f"] \arrow[d] & \mathbb{F}_{2^r} \\ \langle \mathbb{1} \rangle^{\perp} / \langle \mathbb{1} \rangle \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

$$f(x) = \sum_{i=1}^{r} \frac{wt(\phi(\alpha_i x))}{2} \alpha_i^2$$

$$f(x+y) = f(x) + f(y) + (x, y)$$

standard inner product

Type II codes (w.r.t a trace-orthogonal basis B) of length $2n$ over $\mathbb{F}_{2^r}$

$$= \prod_{i=0}^{n-2} (q^i + 1) \qquad q = 2^r$$

Various Type II codes have been constructed.

Pasquier (1980)

$$\exists C \subset \mathbb{F}_8^8 \text{ s.t. } \phi(C) = \text{Golay} \subset \mathbb{F}_2^{24}$$

Mass formula helps classifying Type II codes

Mass formula implies

$$\exists C \subset \mathbb{F}_4^{20} \text{ s.t. } \phi(C) \text{ is exremal} \subset \mathbb{F}_2^{40} \quad \left(\begin{array}{c}\text{i.e.,}\\ \text{min wt}\\ = 8\end{array}\right)$$