

$q = 2^r$ ,  $r =$  positive integer

$u = (u_1, u_2, \dots, u_{2n}) \in \mathbb{F}_q^{2n}$

$v = (v_1, v_2, \dots, v_{2n}) \in \mathbb{F}_q^{2n}$

$(u, v) = \sum_{i=1}^{2n} u_i v_i$

$wt(u) = |\{i \mid u_i \neq 0\}|$

$\mathbb{F}_q^{2n} \supset C$  linear code

$C^\perp = \{u \in \mathbb{F}_q^{2n} \mid (u, v) = 0 \ \forall v \in C\}$

$C$ : self-dual  $\iff C = C^\perp$

For  $q=2$ ,

$C$ : doubly-even  $\iff wt(u) \equiv 0 \pmod{4} \ \forall u \in C$

Doubly-even self-dual (d.e.s.d) codes are also called Type II codes

Gaborit-Pless-Solé-Atkin (1999)

"Type II" codes over  $\mathbb{F}_4$



## Mass Formula

$$\begin{aligned} & \# \text{ d.e.s.d codes of length } 2n \quad (q=2) \\ & = \prod_{i=0}^{n-2} (2^i + 1) \end{aligned}$$

[MacWilliams-Sloane-Thompson (1972), Thompson (1973)]

$$\begin{aligned} & \# \text{ Type II codes of length } 2n \text{ over } \mathbb{F}_4 \\ & = \prod_{i=0}^{n-2} (4^i + 1) \end{aligned}$$

[Gaborit-Pless-Solé-Atkin (1999)]

$$\prod_{i=0}^{n-2} (q^i + 1) = \# \text{ totally singular } (n-1)\text{-spaces}$$

w.r.t. a nondegenerate quadratic form on a  $(2n-2)$ -dim. space over  $\mathbb{F}_q$ , of Witt index  $n-1$ .

[ Segre (1959)  
Ray-Chaudhuri (1962)  
Feng-Dai (1964) ]

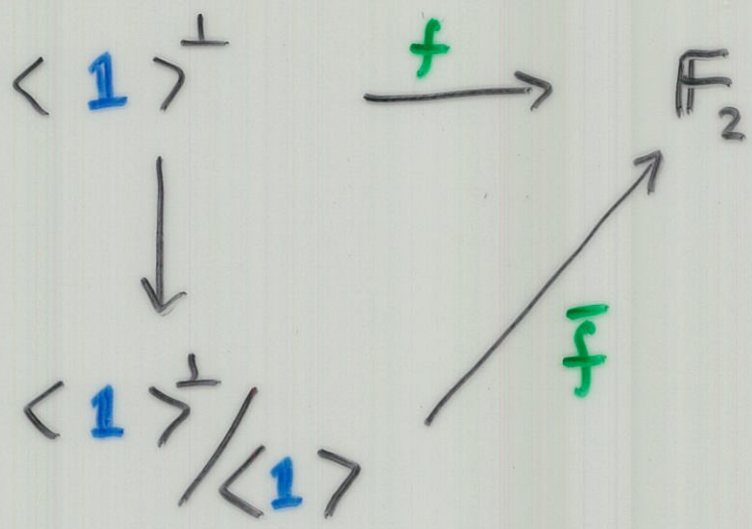
$\mathbf{1}$  = all one vector  $\in \mathbb{F}_2^{2n}$

$$\langle \mathbf{1} \rangle \subset C \subset \langle \mathbf{1} \rangle^\perp$$

d.e.s.d  $\dim C = n$   
 $\updownarrow$  1:1

$$\bar{C} \subset \langle \mathbf{1} \rangle^\perp / \langle \mathbf{1} \rangle$$

totally singular  $(n-1)$ -sp  
 w.r.t. the nondegenerate  
 quadratic form  $\bar{f}$



$$f(x) = \frac{wt(x)}{2}$$

Broué (1977) Puig?



A basis  $B = \{\alpha_1, \alpha_2, \dots, \alpha_r\}$  of  $\mathbb{F}_{2^r}/\mathbb{F}_2$  is called **trace-orthogonal** if (4)

$$\text{Tr}(\alpha_i \alpha_j) = \delta_{ij}$$

$$\mathbb{F}_{2^r} \rightarrow \mathbb{F}_2^r$$

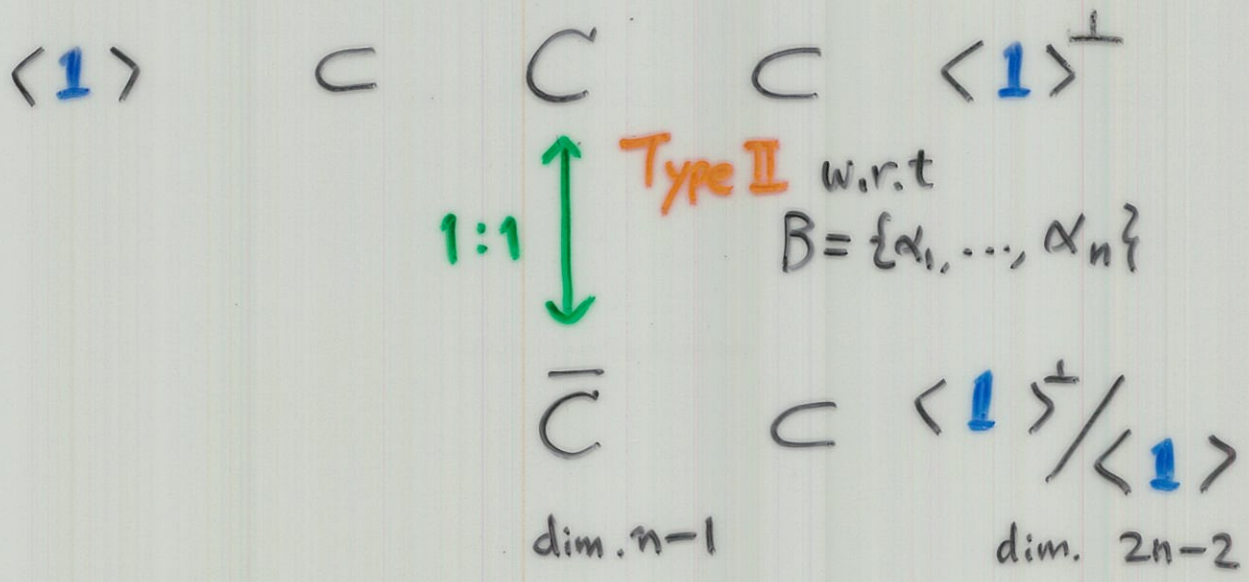
$$x = \sum_{i=1}^r c_i \alpha_i \mapsto (c_1, \dots, c_r)$$

extends to

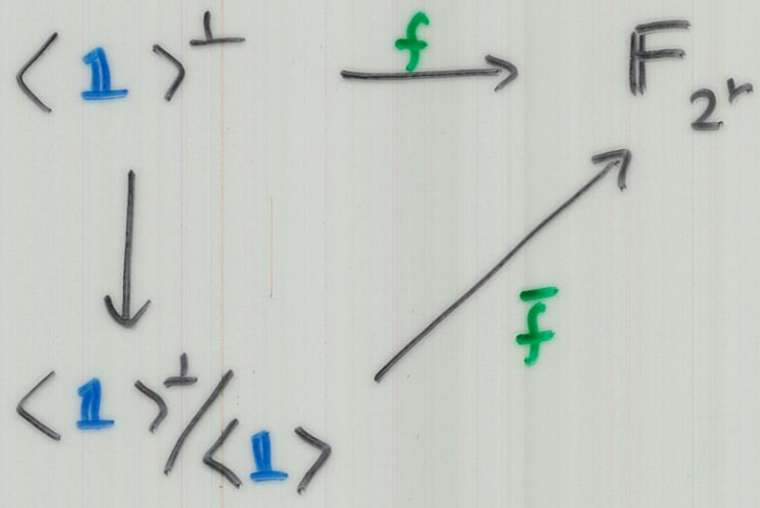
**Gray map**  $\phi: (\mathbb{F}_{2^r})^{2n} \rightarrow \mathbb{F}_2^{2nr}$

**Definition.** A linear code  $C \subset (\mathbb{F}_{2^r})^{2n}$  is

**Type II** (w.r.t  $B$ ) if  $\phi(C)$  is **d.e.s.d.**



totally singular w.r.t the nondegenerate quadratic form  $\bar{f}$



$$f(x) = \sum_{i=1}^r \frac{\text{wt}(\phi(\alpha_i x))}{2} \alpha_i^2$$

$$f(x+y) = f(x) + f(y) + \underset{\substack{\uparrow \\ \text{standard inner product}}}{(x, y)}$$



# **Type II** codes (w.r.t a trace-orthogonal basis  $B$ ) of length  $2n$  over  $\mathbb{F}_{2^r}$

$$= \prod_{i=0}^{n-2} (q^i + 1)$$

$$q = 2^r$$

Various

Type II codes have been constructed.

**Pasquier (1980)**

$$\exists C \subset \mathbb{F}_8^8 \text{ s.t. } \phi(C) = \text{Golay} \subset \mathbb{F}_2^{24}$$

Mass formula helps classifying Type II codes

Mass formula implies

$$\exists C \subset \mathbb{F}_4^{20} \text{ s.t. } \phi(C) \text{ is } \text{extremal} \\ \subset \mathbb{F}_2^{40} \quad \left( \begin{array}{l} \text{i.e.,} \\ \text{min wt} \\ = 8 \end{array} \right)$$