Definition.

A Bose–Mesner (BM) algebra $\mathcal{A}$ is a commutative subalgebra of $M_X(\mathbf{C})$ indexed by a finite set X, satisfying

$$\mathcal{A} \ni I, J$$

and closed under

- entrywise product (denoted by $\circ$)
- transposition

$\mathcal{A}$ has two bases:

- adjacency matrices $A_0, A_1, \dots A_d$ satisfying $A_i \circ A_j = \delta_{ij} A_i$.
- primitive idempotents $E_0, E_1, \dots, E_d$ satisfying $E_i E_j = \delta_{ij} E_i$.

Definition.

A $\mathbf{C}$-linear bijection $\tau : \mathcal{A} \longrightarrow \mathcal{A}$ is an automorphism of $\mathcal{A}$ if

$$\begin{aligned} \tau(AB) &= \tau(A)\tau(B) \\ \tau(A \circ B) &= \tau(A) \circ \tau(B) \\ \tau(A^T) &= \tau(A)^T \end{aligned}$$

for all $A, B \in \mathcal{A}$.

$\mathcal{A}$ is bipartite if

$$\begin{aligned} X &= X_1 \cup X_2, \qquad |X_1| = |X_2| \\ \mathcal{A} &= \mathcal{A}_0 \oplus \mathcal{A}_1 \end{aligned}$$

where

$$\mathcal{A}_0 = \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\} \qquad \mathcal{A}_1 = \left\{ \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \right\}$$

Theorem.

$\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$: bipartite BM algebra

τ: automorphism of $\mathcal{A}$

Assume $\tau^2 = 1$, $\tau|_{\mathcal{A}_0} = 1_{\mathcal{A}_0}$.

Define $\sigma : \mathcal{A} \longrightarrow M_X(\mathbf{C})$ by

$$\sigma(A) = \begin{pmatrix} A_{11} & A_{12} \\ \tau(A)_{21} & A_{22} \end{pmatrix}$$

Then $\sigma(\mathcal{A})$ is a BM algebra.

Note. $\sigma(A \circ B) = \sigma(A) \circ \sigma(B)$ but in general $\sigma(AB) \neq \sigma(A)\sigma(B)$.

Remark.

$\sigma(\mathcal{A}) = \mathcal{A}_0 \oplus \hat{\mathcal{A}}_1$: bipartite BM algebra

$\sigma\tau\sigma^{-1}$: automorphism of $\sigma(\mathcal{A})$

$(\sigma\tau\sigma^{-1})^2 = 1$, $\sigma\tau\sigma^{-1}|_{\mathcal{A}_0} = 1|_{\mathcal{A}_0}$.

Thus by Theorem, we can define

$$\sigma' : \sigma(\mathcal{A}) \longrightarrow M_X(\mathbf{C})$$

Then $\sigma'\sigma(\mathcal{A}) = \mathcal{A}$. In fact,

$$\sigma'\sigma = 1_{\mathcal{A}}, \qquad \sigma\sigma' = 1_{\sigma(\mathcal{A})}.$$

Example.

H: Hadamard matrix.

$$A=\frac{1}{2}\begin{pmatrix} 0 & 0 & J+H & J-H \\ 0 & 0 & J-H & J+H \\ J-H^T & J+H^T & 0 & 0 \\ J+H^T & J-H^T & 0 & 0 \end{pmatrix}$$

$\mathcal{A}=\langle A\rangle$

$\tau=$ transposition

Then $\sigma(\mathcal{A})$ is a BM algebra consisting of symmetric matrices.

$\sigma : \mathcal{A} \longrightarrow \sigma(\mathcal{A})$ preserves entrywise product.

$\{\sigma(A_i)\} =$ adjacency matrices of $\sigma(\mathcal{A})$.

Define $\rho : \mathcal{A} \longrightarrow \sigma(\mathcal{A})$ by

$$\rho(A) = \frac{1+\sqrt{-1}}{2}\sigma(A) + \frac{1-\sqrt{-1}}{2}\sigma\tau(A).$$

Then ρ preserves ordinary product.

$\{\rho(E_i)\} =$ primitive idempotents of $\sigma(\mathcal{A})$.

This leads to the determination of the character table of $\sigma(\mathcal{A})$.

Example.

$H = (1)$: Hadamard matrix

$\mathcal{A} = \langle A \rangle \cong \mathbf{C}[\mathbf{Z}_4]$

$\tau =$ transposition $=$ inversion

Then

$\sigma =$ Gray map

$\sigma(\mathcal{A}) \cong \mathbf{C}[\mathbf{Z}_2 \times \mathbf{Z}_2]$.