The Association Scheme of Injections

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Let $m \leq n$ be positive integers. Let H be the diagonal subgroup of $S_m \times S_m$ regarded as a subgroup of $S_m \times S_n$. Then $H \times S_{n-m}$ can naturally be regarded as a subgroup of $S_m \times S_n$. The permutation character of $S_m \times S_n$ on the cosets of the subgroup $H \times S_{n-m}$ decomposes as follows.

$$\begin{aligned}
1_{H\times S_{n-m}}^{S_m\times S_n} &= (1_{H\times S_{n-m}}^{S_m\times S_m\times S_{n-m}})^{S_m\times S_n} \\
&= \bigoplus_{\chi\in\mathrm{Irr}(S_m)} ((\chi\otimes\chi)\otimes 1_{S_{n-m}})^{S_m\times S_n} \\
&= \bigoplus_{\chi\in\mathrm{Irr}(S_m)} \chi\otimes(\chi\otimes 1_{S_{n-m}})^{S_n}.
\end{aligned}$$

This permutation representation is multiplicity-free. Indeed, by [2, Theorem 2.8.2], we have

$$1_{H\times S_{n-m}}^{S_m\times S_n} = \bigoplus_{\alpha}\bigoplus_{\gamma} \chi_\alpha\otimes\chi_\gamma,$$

where the first sum is over all partitions α of m, and the second sum is over all partitions γ of n satisfying the condition

$$\gamma_1 \geq \alpha_1 \geq \gamma_2 \geq \alpha_2 \geq \cdots . \tag{1}$$

Taking the conjugate partitions, this condition is equivalent to

$$\gamma'_i = \alpha'_i \text{ or } \alpha'_i + 1,$$

or equivalently, $\gamma - \alpha$ is a vertical strip. As we shall see later, the set of pairs of such partitions can be shown to be in one-to-one correspondence with the isomorphism type of the graphs arising from orbitals. In fact, one can associate with such a pair (γ', α'), the disjoint union of graphs with $\gamma'_i + \alpha'_i$ edges which is either a cycle or a path, depending on $\gamma'_i = \alpha'_i$ or $\alpha'_i + 1$. All of these observations reduce to the well-known theory of characters of symmetric groups when $m = n$.

Now let $|M| = m$, $|N| = n$, and let X be the set of all injections from M to N. Then $S_m \times S_n$ acts on X by

$$(\sigma,\tau)\psi := \tau\psi\sigma^{-1}, \qquad \sigma\in S_m,\ \tau\in S_n,\ \psi\in X.$$

Elements of X can be regarded as maximal matchings of the complete bipartite graph $K_{m,n}$. For $\psi, \phi \in X$, we denote by $\Gamma_{\psi,\phi}$ the edge-colored bipartite graph on $M \cup N$ defined as follows: $(x, y) \in M \times N$ is an edge with color 1 (resp. 2) iff $\psi(x) = y$ (resp. $\phi(x) = y$). Forgetting colors in $\Gamma_{\psi,\phi}$ yields a graph $\Delta_{\psi,\phi}$. The graph $\Delta_{\psi,\phi}$ may have multiple edges. Every vertex in M has degree 2 in $\Delta_{\psi,\phi}$, while vertices in N have degree $0, 1$ or 2 in $\Delta_{\psi,\phi}$.

Theorem 1. *Let $\psi, \psi', \phi, \phi' \in X$. Then there exists $(\sigma, \tau) \in S_m \times S_n$ such that $(\sigma, \tau)\psi = \psi'$ and $(\sigma, \tau)\phi = \phi'$ if and only if $\Delta_{\psi,\phi} \cong \Delta_{\psi',\phi'}$.*

Proof. It is clear that if $(\sigma, \tau)\psi = \psi'$ and $(\sigma, \tau)\phi = \phi'$, then $\sigma \times \tau : M \times N \to M \times N$ is an isomorphism from $\Delta_{\psi,\phi}$ to $\Delta_{\psi',\phi'}$. To prove the converse, first observe that thre exists an isomorphism $\Delta_{\psi,\phi} \cong \Delta_{\psi',\phi'}$, leaving the bipartition invariant. Let $\sigma \times \tau$ be such an isomorphism. Let C be a connected component of $\Delta_{\psi,\phi}$. The restriction of $\sigma \times \tau$ to C may not preserve the edge-coloring, but we claim that $\sigma \times \tau$ can be taken in such a way that it preserves the edge-coloring. If C is an isolated vertex in N, then there is nothing to do. If C has an edge, then C is either a cycle or a path, in which edges are colored alternately. If C is a path and $\sigma \times \tau|_C$ does not preserve the edge-coloring, then reflecting the image yields a color-preserving isomorphism. One can argue in a similar manner for cycles, and one obtains a color-preserving isomorphism. Then we have $(\sigma, \tau)\psi = \psi'$ and $(\sigma, \tau)\phi = \phi'$. □

Notice that the orbits of $S_m \times S_n$ are self-paired, since $\Delta_{\psi,\phi} \cong \Delta_{\phi,\psi}$. This gives another proof that the permutation character is multiplicity-free.

Let x_{ij} denote the function on X defined by

$$x_{ij}(\psi) = \begin{cases} 1 & \text{if } \psi(i) = j, \\ 0 & \text{otherwise.} \end{cases}$$

Every complex-valued function on X can be expressed in terms of a polynomial in x_{ij} $(i \in M,\ j \in N)$. Indeed, the characteristic function of $\{\psi\}$ is given by $\prod_{i \in M} x_{i,\psi(i)}$. Every monomial of degree greater than m is zero when regarded as a function on X. We wish to decompose the space of polynomial functions on X into irreducible submodules. In particular, we are concerned with the function space spanned by the polynomials of the form

$$\sum_{\tau: T \to T'} \prod_{i \in T} x_{i,\tau(i)} \tag{2}$$

where T is a t-element subset of M, T' is a t-element subset of N, the sum is taken over all bijections $\tau : T \to T'$. A motivation of doing this is an algebraic approach to the combinatorial object called perpendicular arrays.

Definition 1. Let X be the set of all injections from M to N, where M, N are finite sets. A perpendicular array of strength t is a subset Y of X satisfying the following property: there exists a positive integer λ such that

$$|\{\psi \in Y \mid \psi(T) = T'\}| = \lambda, \tag{3}$$

for any $T \subset M$, $T' \subset N$ with $|T| = |T'| = t$.

The definition looks very similar to that of orthogonal arrays. Much less is known in the theory of perpendicular arrays than in the theory of orthogonal arrays (see [1]).

In terms of the polynomial function (2), the condition (3) can be written as

$$\sum_{\psi \in Y} \sum_{\tau : T \to T'} \prod_{i \in T} x_{i,\tau(i)}(\psi) = \lambda.$$

In the following, we give an irreducible decomposition of the space of polynomial functions of the form (2) when $|T| = |T'| = 1$. It would be nice if we can obtain an irreducible decomposition for arbitrary values of $t = |T| = |T'|$.

Let $E = \langle e_1, \ldots, e_m \rangle$, $F = \langle f_1, \ldots, f_n \rangle$ be the permutation modules for S_m, S_n, respectively. Let $H = \langle x_{ij} | i \in M,\ j \in N \rangle$. Then there is a surjection $E \otimes F \to H$ defined by $e_i \otimes f_j \mapsto x_{ij}$ which commutes with the action of $S_m \times S_n$. The module $E \otimes F$ decomposes as an $S_m \times S_n$-module:

$$\begin{aligned} E \otimes F = &\langle (\sum_{i \in M} e_i) \otimes (\sum_{j \in N} f_j) \rangle \\ &\oplus \langle (\sum_{i \in M} e_i) \otimes (f_1 - f_j | j \in N \rangle \\ &\oplus \langle (e_1 - e_i) \otimes (\sum_{j \in N} f_j) | i \in M \rangle \\ &\oplus \langle (e_1 - e_i) \otimes (f_1 - f_j) | i \in M,\ j \in N \rangle. \end{aligned}$$

Note that each of the four submodules are irreducible. Note also that

$$(e_1 - e_i) \otimes (\sum_{j \in N} f_j) \mapsto \sum_{j \in N} x_{1j} - \sum_{j \in N} x_{ij} = 0$$

while

$$\begin{gathered} (\sum_{i \in M} e_i) \otimes (f_1 - f_j) \mapsto \sum_{i \in M} x_{i1} - \sum_{i \in M} x_{ij}, \\ (e_1 - e_2) \otimes (f_1 - f_2) \mapsto x_{11} + x_{22} - x_{12} - x_{21} \neq 0 \end{gathered}$$

since $(x_{11} + x_{22} - x_{12} - x_{21})(\mathbf{1}) = 2$, where $\mathbf{1}$ denotes the mapping $M \to N$ defined by $\mathbf{1}(i) = i$ for $i \in M$. Observe $\sum_{i \in M} x_{i1} - \sum_{i \in M} x_{ij} = 0$ if and only if $m = n$. Therefore

$$H = \langle \sum_{i \in M} \sum_{j \in N} x_{ij} \rangle \oplus \langle x_{11} + x_{ij} - x_{1j} - x_{i1} | i \in M,\ j \in N \rangle$$

if $m = n$,

$$\begin{aligned} H = &\langle \sum_{i \in M} \sum_{j \in N} x_{ij} \rangle \oplus \langle x_{11} + x_{ij} - x_{1j} - x_{i1} | i \in M,\ j \in N \rangle \\ &\oplus \langle \sum_{i \in M} (x_{i1} - x_{ij}) | j \in N \rangle \end{aligned}$$

if $m < n$. In particular,

$$\dim H = \begin{cases} 1 + (m-1)^2 & \text{if } m = n, \\ mn - m + 1 & \text{if } m < n. \end{cases}$$

For $\psi, \phi \in X$, define $\rho(\psi, \phi)$ by

$$\rho(\psi, \phi) = |\{i \in M | \psi(i) = \phi(i)\}|.$$

Note that if we regard X as a subset of the Cartesian power N^M, then ρ is the restriction of the Hamming distance in N^M to X. Then (X, ρ) becomes a spherical polynomial space in the sense of Conder and Godsil [3].

References

[1] J. Bierbrauer and Y. Edel, Theory of perpendicular arrays. J. Combin. Des. 2 (1994), no. 6, 375–406.

[2] G. James and A. Kerber, The Representation Theory of the Symmetric Group Addison-Wesley, 1981.

[3] M. Conder and C. D. Godsil, The symmetric group as a polynomial space, Combinatorial and graph-theoretical problems in linear algebra (Minneapolis, MN, 1991), 219–227, IMA Vol. Math. Appl., 50, Springer, New York, 1993.