# Self-Orthogonal Designs

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## 1 Introduction

It is well-known that for any positive integer $t$, a nontrivial $t$-$(v, k, \lambda)$ design exists for some $v, k, \lambda$. However, it seems that there are very few self-orthogonal $t$-designs known for large $t$. Recall that a $t$-$(v, k, \lambda)$ design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is said to be self-orthogonal if the parity of the size of intersection of any two blocks is the same as the parity of $k$.

The purpose of this talk is to formulate a conjecture on the nonexistence of self-orthogonal designs for large $t$ with $t \geq [\frac{k}{4}] + 1$, where $k$ is even. We show that the conjecture is true for $(t, k) = (6, 20)$, for example. The method employed is the same as the one developed in [1], where a self-orthogonal 5-$(72, 36, 78)$ design is investigated.

## 2 Saturated designs

**Definition 1.** Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a self-orthogonal $t$-$(v, k, \lambda)$ design. Assume that $k$ is even, so that the binary code $C$ generated by the rows of the block-point incidence matrix of $\mathcal{D}$ is self-orthogonal. We call the design $\mathcal{D}$ saturated, if $C$ is self-dual, $C$ has minimum weight $k$, and every minimum weight codeword of $C$ is the support of a block of $\mathcal{D}$.

Let $\boldsymbol{k} = (k_1, \ldots, k_t)$ be a nonzero left null vector of the $t \times (t-1)$ matrix

$$A_t = \begin{pmatrix} 2 & 4 & \cdots & 2(t-1) \\ \binom{2}{2} & \binom{4}{2} & \cdots & \binom{2(t-1)}{2} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{2}{t} & \binom{4}{t} & \cdots & \binom{2(t-1)}{t} \end{pmatrix} \tag{1}$$

Since the matrix $A_t$ has rank $t-1$, the vector $\boldsymbol{k}$ is unique up to a scalar multiple.

**Lemma 1.**

$$\sum_{i=1}^{t} i(-2)^{i-1}\binom{2t-i-1}{t-1}\binom{2j}{i} = (-1)^{t-1}2^{2t-1}t\binom{j}{j-t}.$$

*Proof.* Induction on $j$. □

**Proposition 2.** *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a self-orthogonal $t$-$(v,k,\lambda)$ design. Assume that $k$ is even, so that the binary code $C$ generated by the rows of the block-point incidence matrix of $\mathcal{D}$ is self-orthogonal. If $C$ has minimum weight $k$ and*

$$t \geq t_0 := [\frac{k}{4}] + 1, \tag{2}$$

*then*

$$\lambda_s = \frac{(-1)^{t_0-1}2^{2t_0-1}t_0\binom{k/2}{k/2-t_0}}{\sum_{i=1}^{t_0} i(-2)^{i-1}\binom{2t_0-i-1}{t_0-1}\binom{k}{i}\prod_{j=i}^{t_0-1}\frac{v-j}{k-j}} \prod_{j=s}^{t_0-1}\frac{v-j}{k-j} \tag{3}$$

*is an integer for $s = 0, 1, \ldots, t_0$.*

*Proof.* Fix a block $B_0$ of $\mathcal{D}$. Since $X$ is a $t_0$-design, we have

$$\sum_{B\in\mathcal{B}} \binom{|B\cap B_0|}{i} = \lambda_i\binom{k}{i} \quad (i = 1, 2, \ldots, t_0), \tag{4}$$

where

$$\lambda_i = \lambda_{t_0}\prod_{j=i}^{t_0-1}\frac{v-j}{k-j}. \tag{5}$$

Put

$$n_j = |\{B \in \mathcal{B} \mid 2j = |B \cap B_0|\}| \quad (j = 0, 1, 2, \ldots).$$

Since $C$ has minimum weight $k$, $|B \cap B_0| \le k/2$ unless $B = B_0$. Thus

$$n_j = 0 \quad \text{for } j > [\frac{k}{4}],\ j \ne \frac{k}{2},$$

and obviously $n_{k/2} = 1$. Now (4) can be written as

$$\sum_{j=0}^{t_0-1} \binom{2j}{i} n_j = (\lambda_i - 1)\binom{k}{i} \quad (i = 1, 2, \ldots, t_0). \tag{6}$$

Let $\boldsymbol{k} = (k_1, \ldots, k_{t_0})$ be the vector defined by

$$k_i = i(-2)^{i-1}\binom{2t_0 - i - 1}{t_0 - 1}. \tag{7}$$

Then $\boldsymbol{k}$ is a left null vector of $A_{t_0}$ by Lemma 1, and hence we have

$$\sum_{i=1}^{t_0} k_i \lambda_i \binom{k}{i} = \sum_{i=1}^{t_0} k_i \binom{k}{i}. \tag{8}$$

By (6) we have

$$\lambda_{t_0} \sum_{i=1}^{t_0} k_i \binom{k}{i} \prod_{j=i}^{t_0-1} \frac{v-j}{k-j} = \sum_{i=1}^{t_0} k_i \binom{k}{i}.$$

Applying (5) again, we obtain

$$\lambda_s = \frac{\sum_{i=1}^{t_0} k_i \binom{k}{i}}{\sum_{i=1}^{t_0} k_i \binom{k}{i} \prod_{j=i}^{t_0-1} \frac{v-j}{k-j}} \prod_{j=s}^{t_0-1} \frac{v-j}{k-j}$$

The result then follows from (7) and Lemma 1. □

As a special case of (3), we have

$$\lambda_{t_0} = \frac{(-1)^{t_0-1} 2^{2t_0-1} t_0 \binom{k/2}{k/2-t_0}}{\sum_{i=1}^{t_0} i(-2)^{i-1} \binom{2t_0-i-1}{t_0-1} \binom{k}{i} \prod_{j=i}^{t_0-1} \frac{v-j}{k-j}}. \tag{9}$$

Observe that the denominator is a polynomial in $v$ of degree $t_0 - 1$ with positive leading coefficient. Thus, for a given $t_0$, there are only finitely many $v$ for which $\lambda_{t_0}$ is an integer.

**Proposition 3.** *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a self-orthogonal $t$-$(v, k, \lambda)$ design, where $k < v$. Assume that $k$ is even, so that the binary code $C$ generated by the rows of the block-point incidence matrix of $\mathcal{D}$ is self-orthogonal. If $C$ has minimum weight $k$ and*

$$t \geq t_0 := [\frac{k}{4}] + 1 \text{ and } k \leq 24, \tag{10}$$

*then*

$$\begin{aligned}(t_0, v, k, \lambda) = (1, v, 2, 1),& \\ &(2,7,4,2),(2,8,4,3),(2,9,4,6),\\ &(2,16,6,2),(2,21,6,4),(2,22,6,5),(2,24,6,10),(2,25,6,20),\\ &(3,16,8,3),(3,22,8,12),(3,23,8,16),(3,24,8,21),\\ &(3,26,8,28),(3,29,8,16),(3,30,8,12),(3,32,8,7),\\ &(3,26,10,3),(3,42,10,9),(3,46,10,8),(3,50,10,6),\\ &(4,47,12,15),(4,48,12,36),(4,51,12,2640),\\ &(5,56,16,42),(5,64,16,91),(5,72,16,78),\\ &(7,120,24,231).\end{aligned}$$

*Proof.* If $k = 2$, then clearly $\lambda = 1$.

If $k = 4$, then

$$\lambda = \frac{6}{10 - v},$$

hence $v = 7, 8$ or $9$.

If $k = 6$, then

$$\lambda = \frac{20}{26 - v},$$

hence $v = 16, 21, 22, 24, 25$.

If $k = 8$, then

$$\begin{aligned}0 &\geq 336(\frac{1}{\lambda} - 1)\\ &= (v - 8)(v - 44),\end{aligned}$$

hence $8 < v \leq 44$. Since $\lambda_3, \ldots, \lambda_0$ are also integers, we have $v = 16, 22, 23, 24, 26, 29, 30$ or $32$.

If $k = 10$, then

$$\begin{aligned} 0 &\geq 1152(\frac{1}{\lambda} - 1) \\ &= (v - 10)(v - 74), \end{aligned}$$

hence $10 < v \leq 74$. Since $\lambda_3, \ldots, \lambda_0$ are also positive integers, we have $v = 10, 26, 42, 46$ or $50$.

If $k = 12$, then

$$\lambda = -\frac{31680}{v^3 - 127v^2 + 5456v - 80592},$$

hence

$$\begin{aligned} 0 &\geq v^3 - 127v^2 + 5456v - 80592 \\ &= v^2(v - 127) + 5456(v - 15) + 1248. \end{aligned}$$

This implies $v < 127$. Since $\lambda_4, \ldots, \lambda_0$ are also positive integers, we have $v = 12, 36, 47, 48, 51, 52$ or $57$.

If $k = 14$, then

$$\lambda = -\frac{549120}{5v^3 - 875v^2 + 52952v - 1132668},$$

hence

$$\begin{aligned} 0 &\geq 5v^3 - 875v^2 + 52952v - 1132668 \\ &= 5v^2(v - 175) + 52950(v - 22) + 32232. \end{aligned}$$

This implies $v \leq 174$. Since $\lambda_4, \ldots, \lambda_0$ are also positive integers, we have $v = 14$.

If $k = 16$, then

$$\lambda = \frac{4193280}{v^4 - 235v^3 + 20960v^2 - 848000v + 13292544},$$

hence,

$$\begin{aligned} 0 &\geq 4193280(\frac{1}{\lambda} - 1) \\ &= (v - 16)(v^2(v - 219) + 17456(v - 33) + 7344). \end{aligned}$$

Thus $v < 219$. Since $\lambda_5, \ldots, \lambda_0$ are also positive integers, we have $v = 16, 56, 64$ or $72$.

If $k = 18$, then

$$\begin{aligned} 0 &\geq 102359040(\frac{1}{\lambda(18)} - 1) \\ &= 7v^3(v-299) + 240404v(v-53) + 37200v + 162256464. \end{aligned}$$

Thus $16 \leq v < 299$. Since $\lambda_5, \ldots, \lambda_0$ are also positive integers, we have $v = 18$.

If $k = 20$, then

$$\begin{aligned} 0 &> -\frac{5000970240}{\lambda} \\ &= 7v^4(v-376) + 399861v^2(v-78) + 1215608048(v-17) \\ &\quad + 350602v^2 + 887460016, \end{aligned}$$

hence $v < 376$. Since $\lambda_6, \ldots, \lambda_0$ are also positive integers, we have $v = 20$.

If $k = 22$, then

$$\begin{aligned} 0 &> -\frac{2500485120}{\lambda} \\ &= v^4(v-456) + 84659v^2(v-96) + 394932588(v-21) \\ &\quad + 79328v^2 + 200001388, \end{aligned}$$

hence $v < 456$. Since $\lambda_6, \ldots, \lambda_0$ are also positive integers, we have $v = 22$.

If $k = 24$, then

$$\begin{aligned} 0 &\geq 148852408320(\frac{1}{\lambda} - 1) \\ &= (v-24)((v-526)(v^4 + 114511v^2 + 47117192v + 25600739920) \\ &\quad + 13441571438560), \end{aligned}$$

hence $v < 526$. Since $\lambda_7, \ldots, \lambda_0$ are also positive integers, we have $v = 24$ or $120$. □

# 3 Unsaturated designs

In this section we let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a self-orthogonal $t$-$(v, k, \lambda)$ design with $k$ even, so that the binary code $C$ generated by the rows of the block-point incidence matrix of $\mathcal{D}$ is self-orthogonal. We assume that the design $\mathcal{D}$ is unsaturated, i.e., either $C$ is not self-dual, or $C$ has a codeword of weight at most $k$ different from the support of a block of $\mathcal{D}$. This implies that there exists a coset $x+C$, possibly equal to $C$, such that it contains a nonzero vector with minimal weight other than the support of any block of $\mathcal{D}$. Let $S$ be the support of such a vector, put $m=|S|$. Then

$$|B \cap S| \leq \frac{k}{2}$$

for any block $B$. Since

$$\sum_{B\in\mathcal{B}} \binom{|B\cap S|}{i} = \lambda_i \binom{m}{i} \quad (i=1,2,\ldots,t),$$

putting

$$n_j = |\{B \in \mathcal{B} \mid 2j = |B \cap B_0|\}| \quad (j=0,1,2,\ldots),$$

we have

$$\sum_{j=0}^{[k/4]} \binom{2j}{i} n_j = \lambda_i \binom{m}{i}.$$

Assume

$$t \geq t_0 := [\frac{k}{4}] + 1.$$

Then, as in the previous section, we have

$$\sum_{i=1}^{t_0} i(-2)^{i-1} \binom{2t_0-i-1}{t_0-1} \lambda_i \binom{m}{i} = 0$$

By (5), we have

$$\sum_{i=1}^{t_0} i(-2)^{i-1} \binom{2t_0-i-1}{t_0-1} \binom{m}{i} \prod_{j=i}^{t_0-1} \frac{v-j}{k-j} = 0 \tag{11}$$

If $k$ is given, then this is a Diophantine equation in $m, v$. The only integer solutions of equation (11) in the range $0 < m < v \leq 1000$, $k = 8, 10, \ldots, 20$

are

$$
\begin{aligned}
k = 8,\ (v,m) &= (16,4),(16,6),(22,6),(22,7),(23,7),\\
k = 10,\ (v,m) &= (20,4),(22,6),(26,6),\\
k = 12,\ (v,m) &= (24,8),(36,8),(47,11),(68,16),(156,36),(311,71),\\
k = 14,\ (v,m) &= (80,16),(159,31),\\
k = 16,\ (v,m) &= (32,8),(43,8),(43,11),(48,12),(56,12),(58,13),\\
k = 18,\ (v,m) &= (36,8).
\end{aligned}
$$

Therefore, we obtain the following result.

**Proposition 4.** *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a self-orthogonal $t$-$(v, k, \lambda)$ design. Assume that $k$ is even, and $t \geq [\frac{k}{4}] + 1$, and $8 \leq k \leq 20$, $2k \leq v \leq 1000$. The binary code $C$ generated by the rows of the block-point incidence matrix $M$ of $\mathcal{D}$ is self-dual and the codewords of $C$ of weight $k$ are precisely the rows of $M$, unless $(k, v)$ is one of the pairs listed above, in which case, either $C^{\perp}/C$ contains a coset of weight $m$ whose values are listed above.*

# References


[1] M. Harada, M. Kitazume and A. Munemasa, On a 5-design related to an extremal doubly-even self-dual code of length 72, J. Combin. Theory, Ser. A, 107 (2004), 143–146.