Combinatorial Structures Derived from Extremal Even Unimodular Lattices

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A Cube Approximates a Sphere

A cube Q consisting of 8 vertices

$$\{(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}})\}$$

is contained in the unit sphere S^2 in $\mathbb{R}^3$.

Observe that Q is a good approximation of S^2 in the sense that

$$\frac{1}{8}\sum_{\mathbf{x}\in Q} f(\mathbf{x}) = \frac{1}{4\pi}\int_{S^2} f(\mathbf{x})d\sigma \qquad (1)$$

for any polynomial $f(\mathbf{x}) = f(x, y, z)$ of degree at most 3.
Indeed,

$$f(x, y, z) = ax^3 + by^3 + \cdots + cz + d,$$

the verification of (1) is reduced to the case $f(x, y, z) = x^2$.

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$$\frac{1}{8}\sum_{\mathbf{x}\in Q} x^2 \overset{?}{=} \frac{1}{4\pi}\int_{S^2} x^2 d\sigma$$

But then $LHS = \frac{1}{3} = RHS$, since

$$Q = \{(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}})\}$$

$$\int_{S^2} x^2 d\sigma = \frac{1}{3}\int_{S^2}(x^2+y^2+z^2)d\sigma = \frac{1}{3}\int_{S^2} 1 d\sigma.$$

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Definition of a Spherical Design

Definition

A spherical t-design X is a finite subset of the sphere $S^{n-1}(\mu) \subset \mathbb{R}^n$ of radius $\sqrt{\mu}$ s.t.

$$\frac{1}{|X|}\sum_{\mathbf{x}\in X} f(\mathbf{x}) = \frac{1}{\text{surface area of } S^{n-1}(\mu)}\int_{S^{n-1}(\mu)} f(\mathbf{x})d\sigma$$

holds for any polynomial $f(\mathbf{x}) = f(x_1, \ldots, x_n)$ of degree $\leq t$.

Example

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Spherical Designs

Definition

The strength of a finite subset $X \subset S^{n-1}(\mu)$ is the largest integer t for which X is a spherical t-design.

The degree s of X is the size of the set

$$\{(\mathbf{x}, \mathbf{y}) \mid \mathbf{x}, \mathbf{y} \in X, \ \mathbf{x} \neq \mathbf{y}\}.$$

Example

cube	$t = 3$	$\lvert X\rvert = 8$	$s = 3$
icosahedron	$t = 5$	$\lvert X\rvert = 12$	$s = 3$
root system E_8	$t = 7$	$\lvert X\rvert = 240$	$s = 4$

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Even Unimodular Lattices

Definition

A lattice L of dimension n is a $\mathbb{Z}$-submodule of $\mathbb{R}^n$ generated by a basis of $\mathbb{R}^n$.

- L is integral if $(x, y) \in \mathbb{Z}\ \forall x, y \in L$
- L is unimodular if $\det(\text{Gram matrix}) = 1$
- L is even if $(x, x) \in 2\mathbb{Z}\ \forall x \in L$.

An even unimodular lattice of dimension n exists iff $n \equiv 0 \pmod 8$.

Example

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The Leech Lattice

$\exists$ unique even unimodular lattice L of dimension 24 containing no element of norm 2, that is, $(x, x) \in \{4, 6, 8, \ldots\}$ for $\forall x \in L$, $x \neq 0$.
For a given lattice L and a real number μ, denote by L_μ the set

$$\{x \in L \mid (x, x) = \mu\} \subset S^{n-1}(\mu).$$

Example

For the E_8-lattice L, $|L_2| = 240$, and L_2 is a spherical 7-design.
For the Leech lattice $L_2 = \emptyset$, $|L_4| = 196560$, and L_4 is a spherical 11-design.

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Subconstituents

Let X be a spherical t-design in the unit sphere in $\mathbb{R}^n$, and pick $y \in X$.

A subconstituent of X with respect to y and η is

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Example

The two nontrivial subconstituents of a cube are equilateral triangles.

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The two nontrivial subconstituents of an icosahedron are pentagons.

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Theorem of Delsarte–Goethals–Seidel, 1977

Theorem

Let X be a spherical t-design in the unit sphere in $\mathbb{R}^n$, and pick $y \in X$. Let

$$s' = |\{(x, y) \mid x \in X,\ (x, y) \neq \pm 1\}|.$$

Then every subconstituent of X with respect to y is a $(t + 1 - s')$-design in $\mathbb{R}^{n-1}$.

Example

An icosahedron is a spherical 5-design, and $s' = 2$. Its subconstituents are regular pentagons, and they are $(t + 1 - s') = 4$-design.

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Subconstituents in the Leech Lattice

Example

Let L be the Leech lattice. The sizes of the subconstituents of L_4 are:

$$1 + 4600 + 47104 + 93150 + 47104 + 4600 + 1 = 196560.$$

Each of the nontrivial subconstituents (of sizes 4600, 47104, 93150) is a spherical $(t + 1 - s') = 7$-design.

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Generalized Subconstituents

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Example

A cube has a square as a subconstituent with respect to a normal vector of a face.

Example

Let L be the Leech lattice, $y \in L_6$. Then the subconstituents of $X = L_4$ with respect to y have sizes

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This implies that each of the "generalized" subconstituents of sizes 552, 11178, 48600 and 75900 is a spherical 11+1-7=5-design.

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Another Theorem of Delsarte–Goethals–Seidel

Theorem

If X is a spherical t-design with degree s satisfying $t \geq 2s - 2$, then X carries a (Q-polynomial) association scheme.

Remark

Looks somewhat similar to a theorem of Cohn-Kumar on universal optimality of spherical codes.
There are association schemes related to spherical designs, whose existence is not guaranteed by the above theorem.

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New (?) Association Schemes

Let L be the Leech lattice, $X = L_4$ has $t = 11$, $s' = 5$.

Subconstituents:

$$1 + 4600 + 47104 + 93150 + 47104 + 4600 + 1 = 196560$$

Every nontrivial subconstituent is a spherical 7-design.
4600: $s = 4$, hence association scheme ($t \geq 2s - 2$)
47104: $s = 5$, also association scheme (why?)
Subconstituents of "47104":

$$1 + 2025 + 15400 + 22275 + 7128 + 275 = 47104$$

Every nontrivial subconstituent is a spherical $(7 + 1 - 5) = 3$-design.
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4600: $s = 4$, hence association scheme ($t \geq 2s - 2$)
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Subconstituents of "47104":

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Every nontrivial subconstituent is a spherical $(7 + 1 - 5) = 3$-design.
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Next Interesting Case is Dimension 48

Because 48 is the dimension when the lower bound on the minimum norm of even unimodular lattices jumps from 4 to 6.

The number of the shortest vectors is huge: 52,416,000.

One cannot work directly with the set of shortest vectors.

Besides, there are three lattices known, up to isometry.

Let L be an even unimodular lattice of dimension 48, minimum norm 6 (We wish to classify such lattices, if possible)

Theorem (Venkov, 1984)

L_6 is a spherical 11-design.

Using the property of being a spherical 11-design, we can compute the sizes of generalized subconstituents.

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Subconstituents in Dimension 48

Let L be an even unimodular lattice of dimension 48, minimum norm 6. Then the sizes of subconstituents are:

- w.r.t. elt. of norm 6:
  1, 36848, 1678887, 12608784, 23766960, . . . ,
- w.r.t. elt. of norm 8:
  2256, 192512, 2905728, 12816384, 20582240, . . . ,
- w.r.t. elt. of norm 10:
  100, 17150, 475300, 3898200, 12612600, 18409300, . . . ,
- w.r.t. certain elt. of norm 12:
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Equiangular Lines

Theorem

Let L be an even unimodular lattice of dimension 48, minimum norm 6. Then for every element $\alpha \in L_{10}$,

$$\{\pm(x - \frac{5}{2}\alpha) \mid x \in L_6,\ (x,\alpha) = 5\}$$

is a set of equiangular lines with angle $\arccos \frac{1}{7}$, of size 50. Also, there exists an element $\beta \in L_{14}$ such that

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