On Graphs with Complete Multipartite μ-Graphs

Akihiro Munemasa

Graduate School of Information Sciences
Tohoku University

(joint work with Aleksandar Jurišić and Yuki Tagami)

Pusan National University
February 10, 2007.

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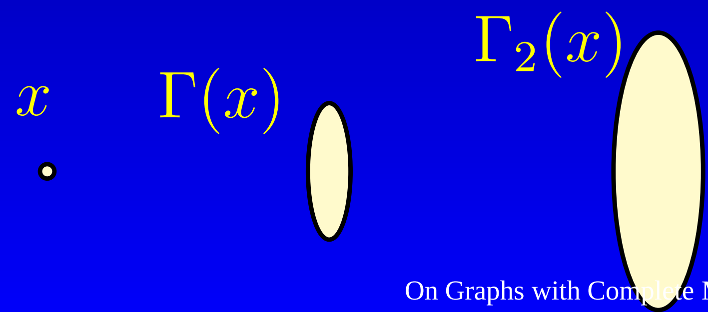

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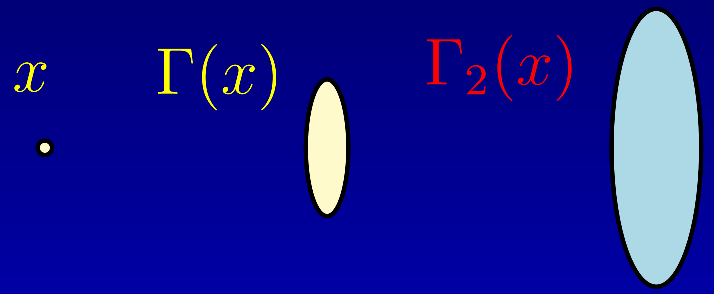

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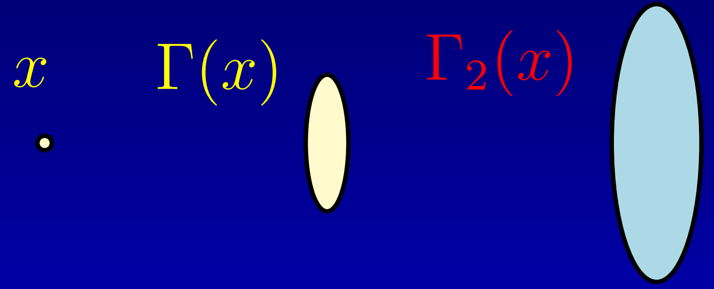

Example: locally 5-gon $\Longrightarrow$ icosahedron

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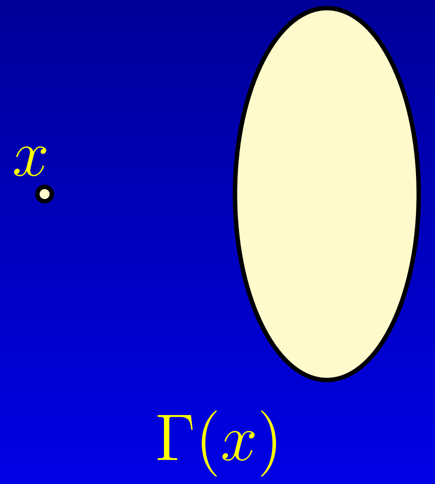

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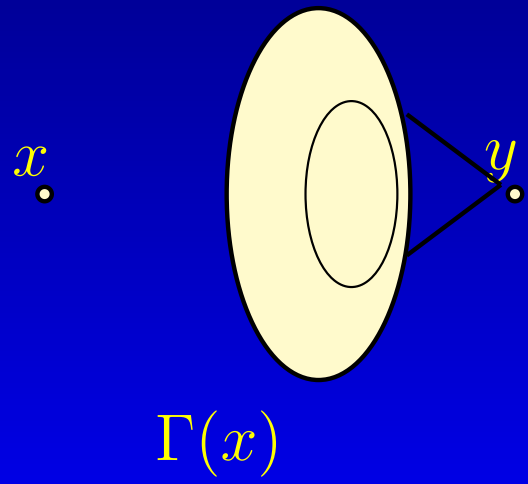

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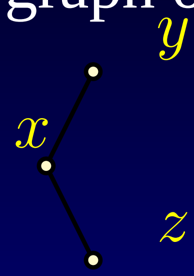

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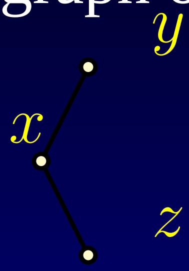

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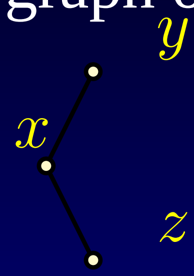

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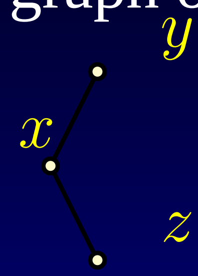

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Lemma. μ, μ' constant *and* $\mu > \mu' + 1 \implies$ regular.

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Γ is a distance-regular graph if

$$p_{ij}^h = |\Gamma_i(x) \cap \Gamma_j(y)| \quad \text{constant}$$

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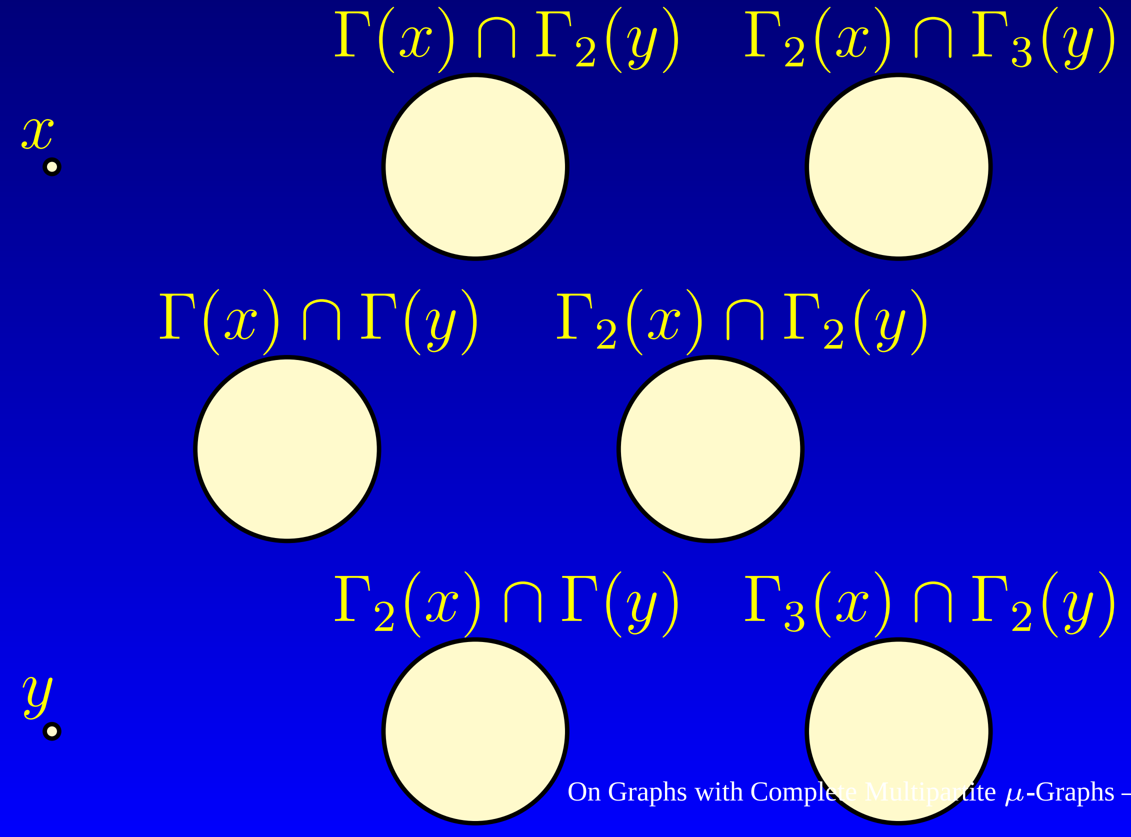

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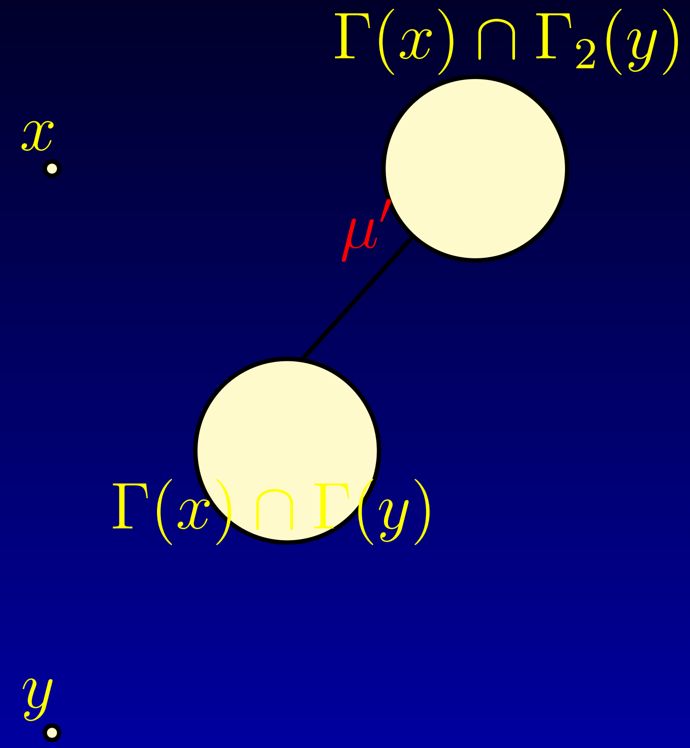

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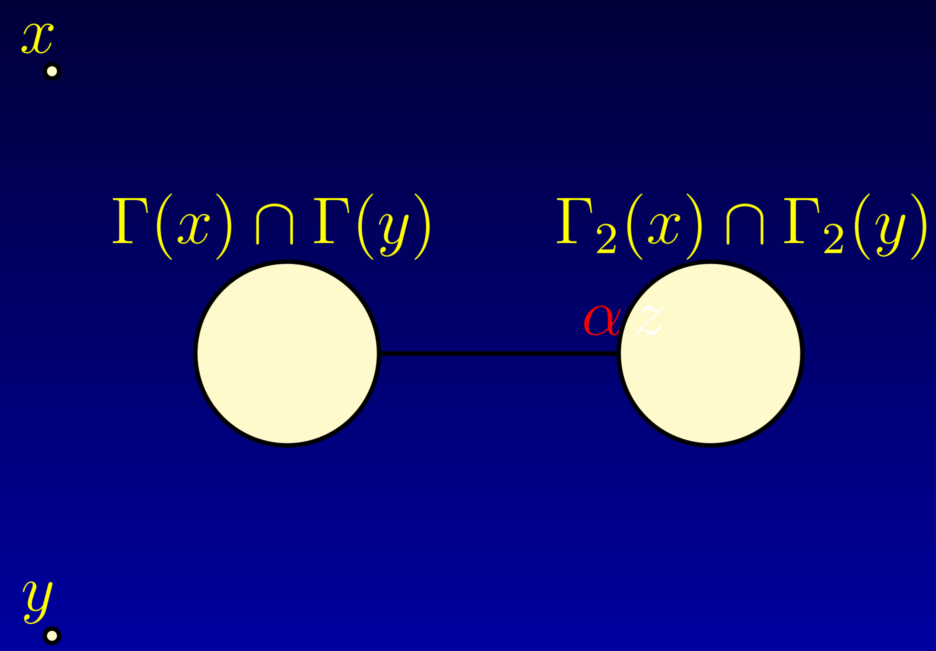

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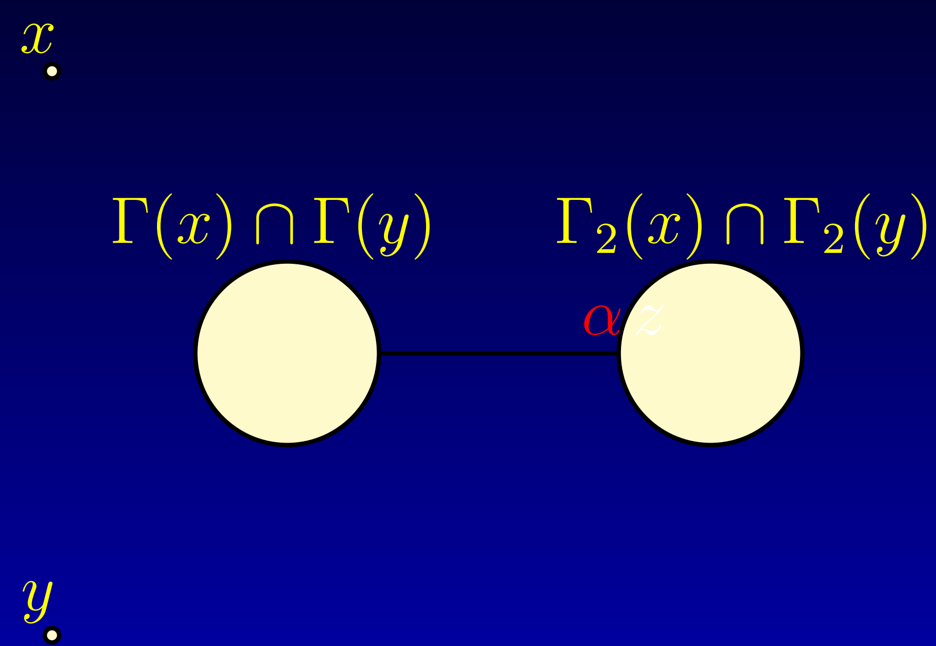

If $|\Gamma(x, y, z)|$ is constant whenever $d(x, y) = 1$, $z \in \Gamma_2(x) \cap \Gamma_2(y)$ (and there exists at least one such triple x, y, z), then we say

α exists

and denote this constant as α.

Examples

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If α exists in Γ, then α exists in a local graph Δ of Γ, and $\alpha(\Delta) = \alpha(\Gamma) - 1$.

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Moreover, $t \leq 4$, with equality holds only if $n = 3$.

Proof

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α	α	$\alpha - 1$	$\cdots$	$\alpha - t + 2$	$\alpha - t + 1$

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