Spherical Designs

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Definitions

A spherical t-design is a finite nonempty subset X of $\mathbb{R}^d$ of constant norm ρ, satisfying

$$\int_{S^{d-1}(\rho)} f(\xi)d\xi = \frac{1}{|X|}\sum_{x\in X} f(x)$$

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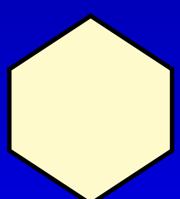
Example. Regular polygon in $\mathbb{R}^2$.



$$\begin{bmatrix} \cos\frac{2\pi}{n} & \cos\frac{2\pi\cdot 2}{n} & \cdots & \cos\frac{2\pi\cdot n}{n} \\ \sin\frac{2\pi}{n} & \sin\frac{2\pi\cdot 2}{n} & \cdots & \sin\frac{2\pi\cdot n}{n} \end{bmatrix}$$

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$m = 1 \implies$ a regular n-gon in $\mathbb{R}^2$.

$n = 2m + 1 \implies$ a regular $2m$-simplex in $\mathbb{R}^{2m}$.

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$$W_{1,n} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 & 1 & -1 & \cdots & 1 & -1 \end{bmatrix}$$

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What if both d and n are odd?

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Theorem. Let n, d be odd positive integers satisfying $n \geq 2d + 1 \geq 7$. Then

$$W_{d,n} = \begin{bmatrix} \alpha W_{d-1,n-d} & W_{d-1,d} \\ \beta W_{1,n-d} & 0 \end{bmatrix}$$

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is a spherical 2-design of n points in $\mathbb{R}^d$, for some $\alpha, \beta \neq 0$ (unique up to sign). Simpler than the original one given by Mimura, 1990.

Existence of spherical 2-designs

A spherical 2-design of n points in $\mathbb{R}^d$ exists if:

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d	n
2	3
3	
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7	

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		n
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d \ n							
2	3	→	→	→	→	→	→
3		4		6		8	10
4			5	→	→	→	→
5				6		8	10
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	2	3	→	→	→	→	→	→	→	→
	3		4		6	7	8	9	10	11
d	4			5	→	→	→	→	→	→
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	3	×	4	×	6	7	8	9	10	11
d	4	×	×	5	→					
	5	×	×	×	6	×	8	9	10	11
	6	×	×	×	×	7	→			
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	⋮									

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$$W_{d,n}(U) = \begin{bmatrix} \alpha W_{d-1,n-d} & UW_{d-1,d} \\ \beta W_{1,n-d} & 0 \end{bmatrix}$$

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This suggests that

there are too many spherical 2-designs

when both d and n are odd and $n \geq 2d + 1 \geq 7$

(observed by Sali, 1993).

Rigidity (Bannai, 1988)

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$$W = \left[\begin{array}{c|c} X_1 & \begin{array}{c} Y \\ \boldsymbol{a}\mathbf{1}^T \end{array} \end{array} \right] \begin{array}{l} \}m \\ \}d-m \end{array}$$

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rank $X_1 = d$,
then W is not rigid.

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Theorem (Sali, 1994).

$$W = \left[\begin{array}{c|c} X_1 & \begin{array}{c} Y \\ \boldsymbol{a}\mathbf{1}^T \end{array} \end{array} \right] \begin{array}{l} \}m \\ \}d-m \end{array} \qquad \boldsymbol{a}\mathbf{1}^T = \begin{bmatrix} a_{m+1} & \cdots & a_{m+1} \\ \vdots & & \vdots \\ a_d & \cdots & a_d \end{bmatrix}$$

W: spherical t-design in $\mathbb{R}^d$,
Y: spherical t-design in $\mathbb{R}^m$, $m \geq 2$,
~~rank $X_1 = d$,~~
then W is not rigid.

Rigidity

Theorem.

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W: spherical t-design in $\mathbb{R}^d$,
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suppose there exists a column vector

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}' \\ \boldsymbol{x}'' \end{bmatrix} \text{ of } \begin{bmatrix} X' \\ X'' \end{bmatrix} \text{ such that } \boldsymbol{x}' \neq 0$$

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(much weaker than requiring $\operatorname{rank} \begin{bmatrix} X' \\ X'' \end{bmatrix} = d$)

Proof

$$W(U) = \left[\begin{array}{c|c} X' & UY \\ X'' & \boldsymbol{a}\mathbf{1}^T \end{array}\right] \qquad U \in SO(m, \mathbb{R})$$

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Want to show: $\forall \varepsilon > 0$, $\exists W(U)$: ε-close to W, yet $W(U) \ncong W$ (under $SO(d, \mathbb{R})$)

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yet $W(U) \not\cong W \iff A(W) \neq A(W(U))$

$$A(W) = \{(\boldsymbol{u}, \boldsymbol{v}) \mid \boldsymbol{u}, \boldsymbol{v} \in W,\ \boldsymbol{u} \neq \boldsymbol{v}\}.$$

is the "angle set" of W.

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Fix a column vector $\boldsymbol{y}$ of Y, ($\boldsymbol{y} \neq 0$ since Y is a spherical design)

$$\boldsymbol{x}^T \begin{bmatrix} \boldsymbol{y} \\ \boldsymbol{a} \end{bmatrix} \in A(W)$$

Proof

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$$\begin{aligned} \phi : SO(m,\mathbb{R}) &\longrightarrow \mathbb{R} \\ U &\longmapsto \boldsymbol{x}^T \begin{bmatrix} U\boldsymbol{y} \\ \boldsymbol{a} \end{bmatrix} \end{aligned}$$

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Since ϕ is an open mapping, $\forall \varepsilon > 0$,

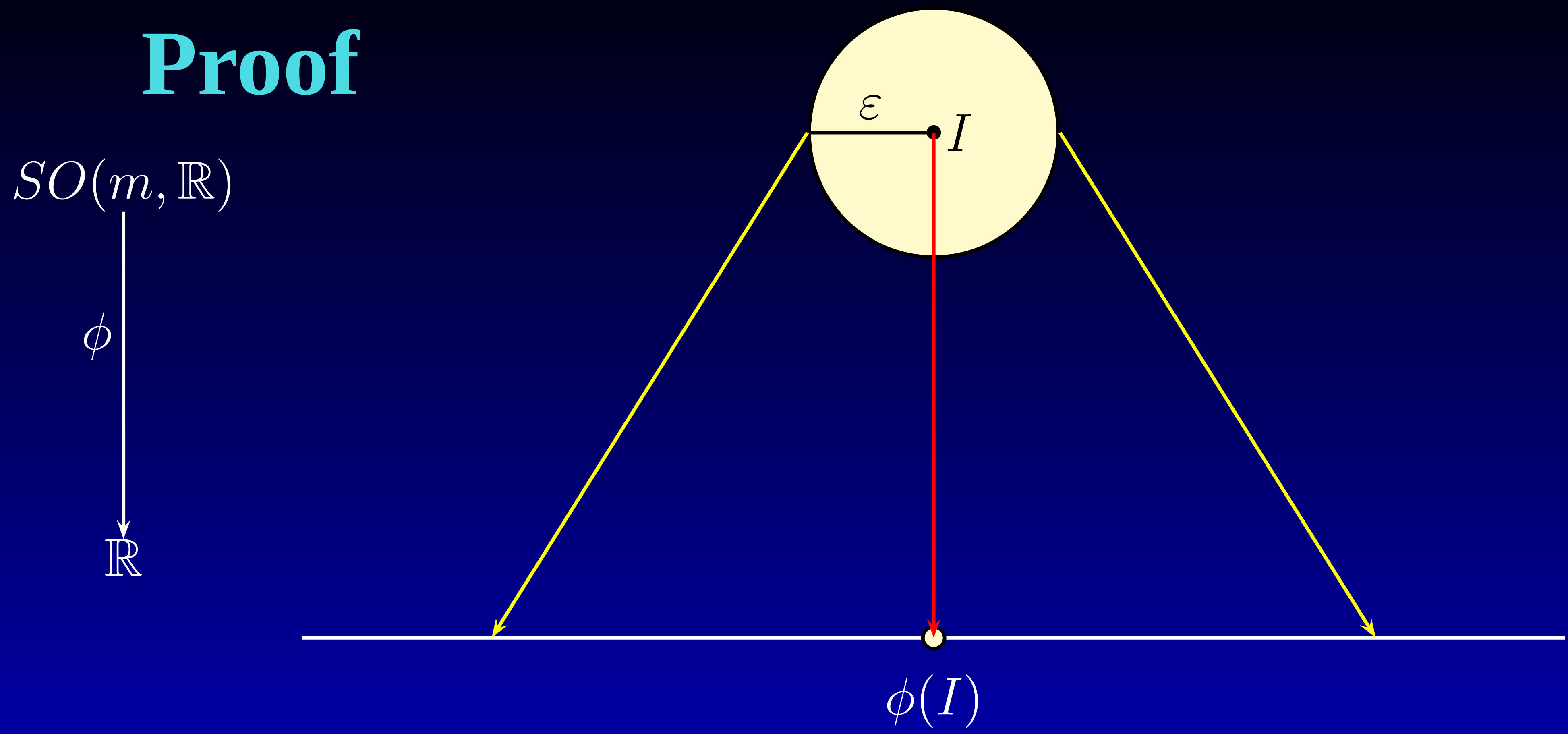
$$A(W) \ni \phi(I) \in \phi(\{U \in SO(m,\mathbb{R}) \mid \|U - I\| < \varepsilon\}) : \text{ open in } \mathbb{R}$$

open neighborhood of I in $SO(m, \mathbb{R})$

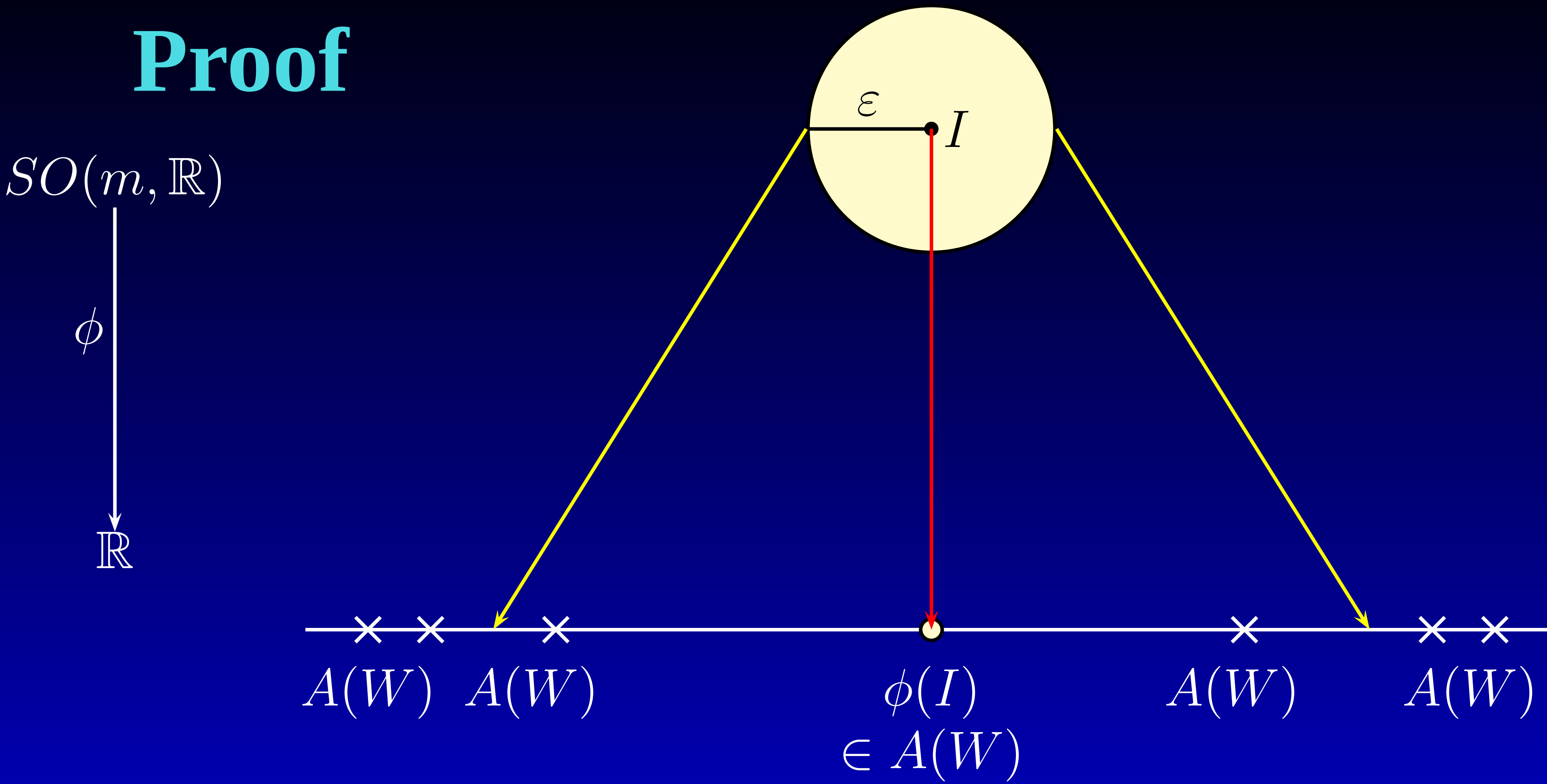
Proof

$$SO(m, \mathbb{R}) \xrightarrow{\ \phi\ } \mathbb{R}$$

Proof



Proof



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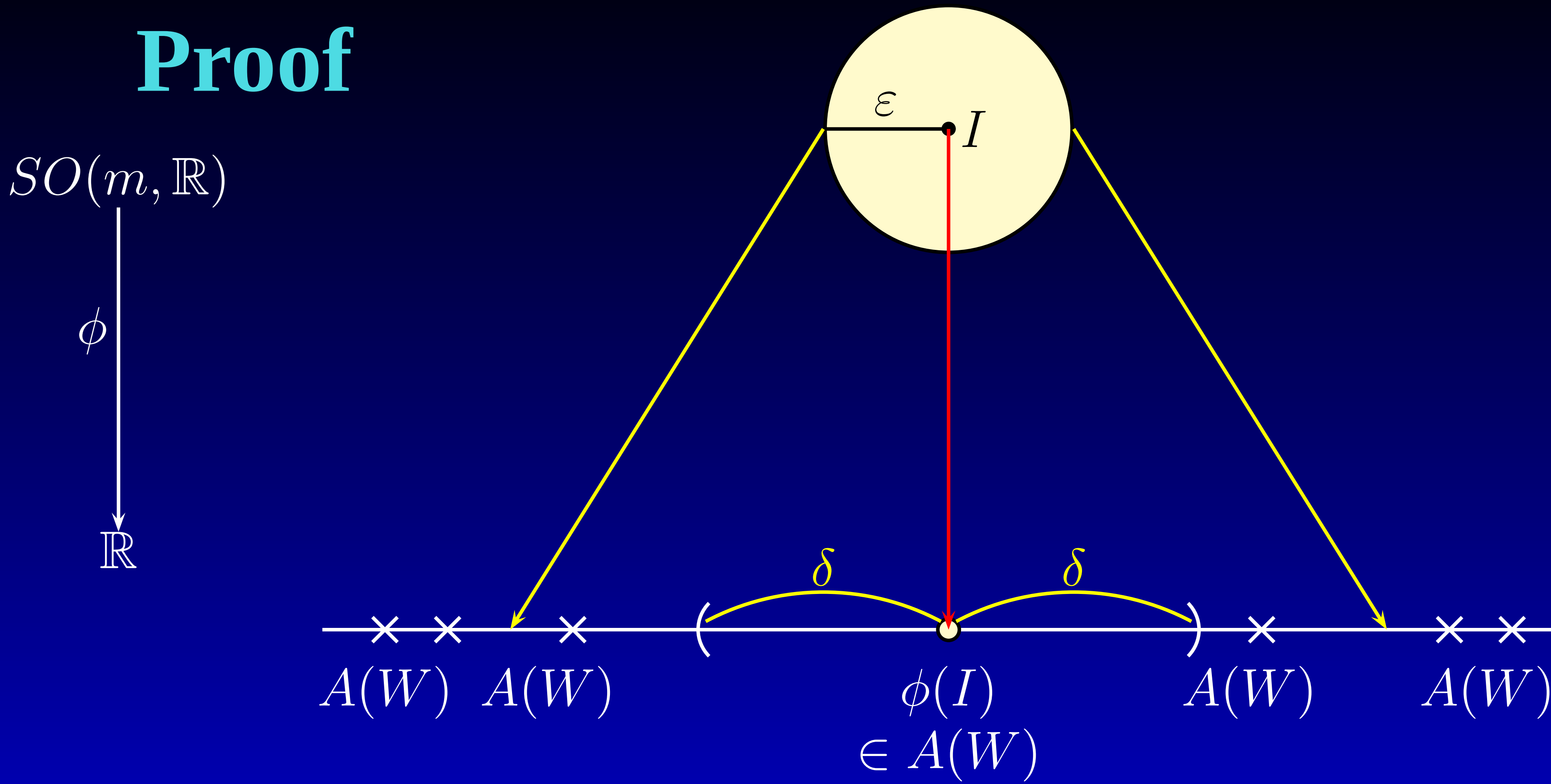

$SO(m,\mathbb{R})$
ϕ
$\mathbb{R}$
ε
I
δ
δ
$A(W)$
$A(W)$
$\phi(I)$
$\in A(W)$
$A(W)$
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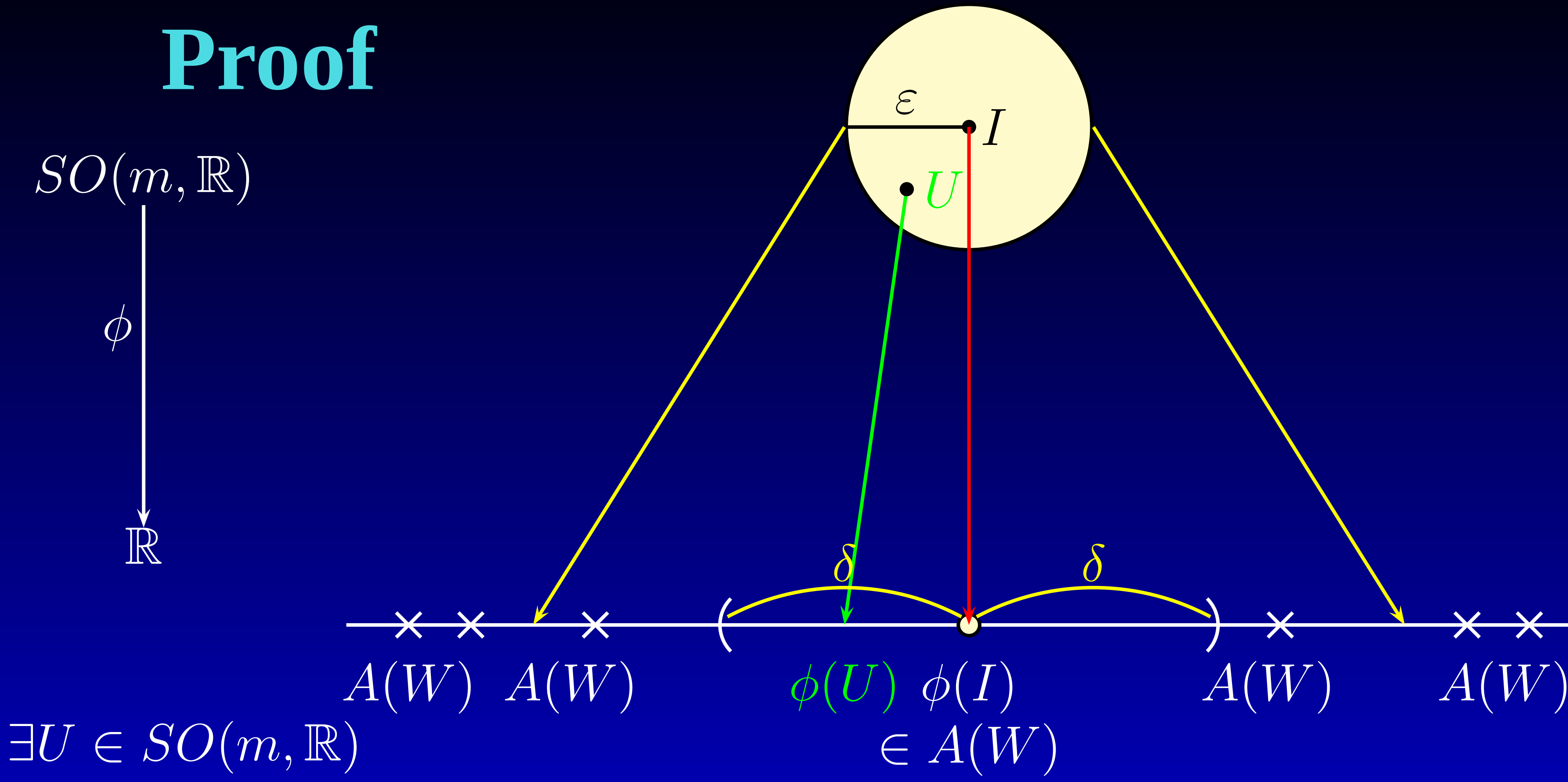

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ϕ
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$A(W)$ $A(W)$
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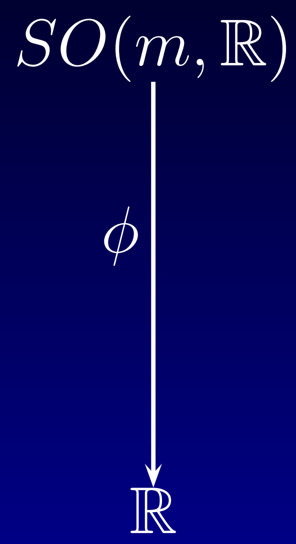
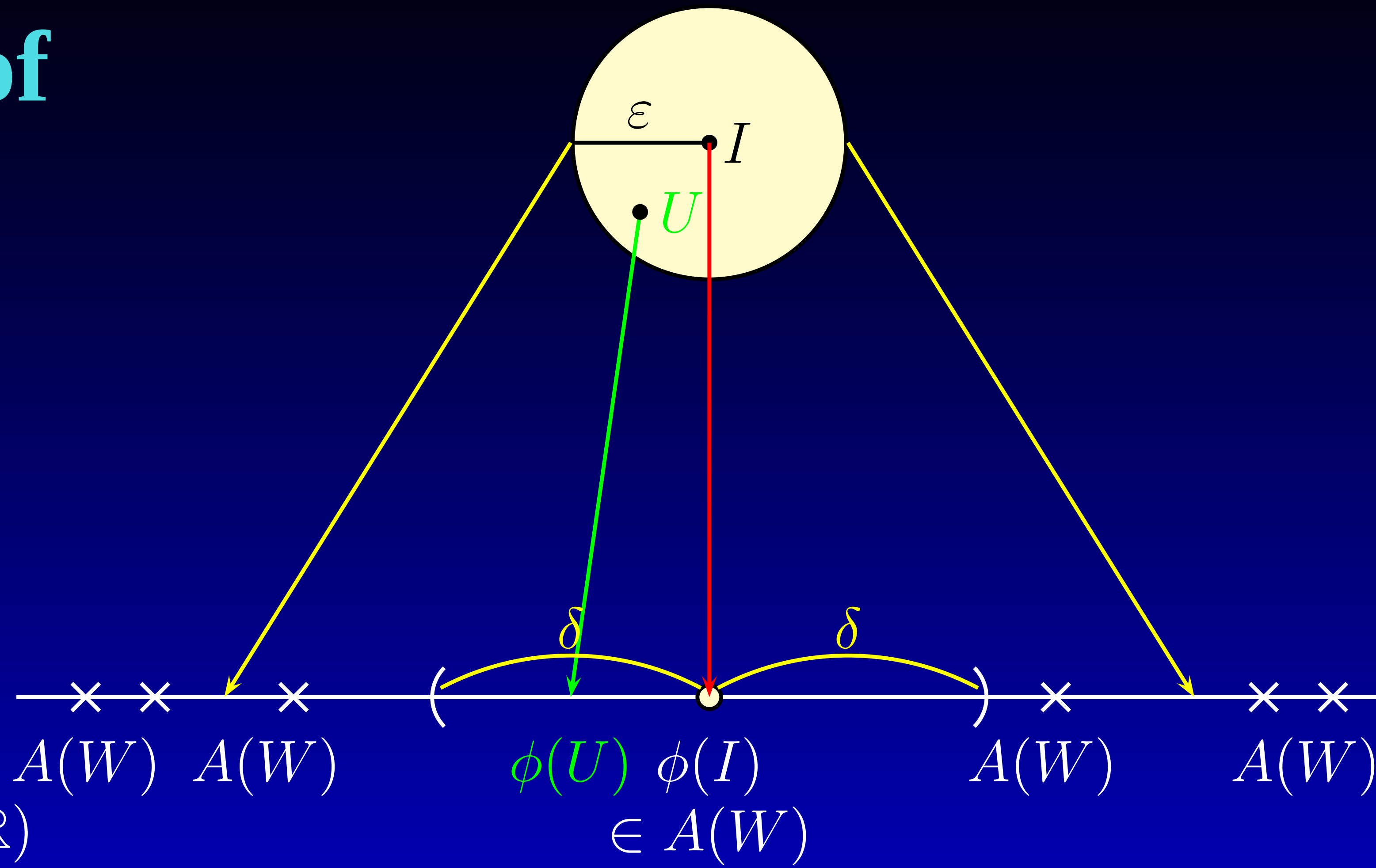
Proof



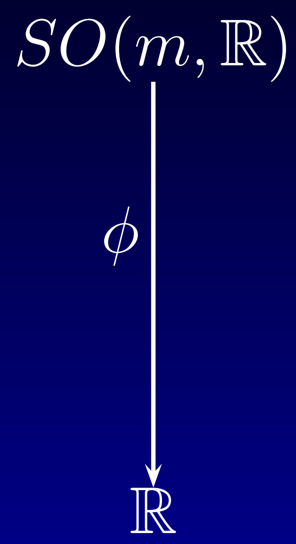


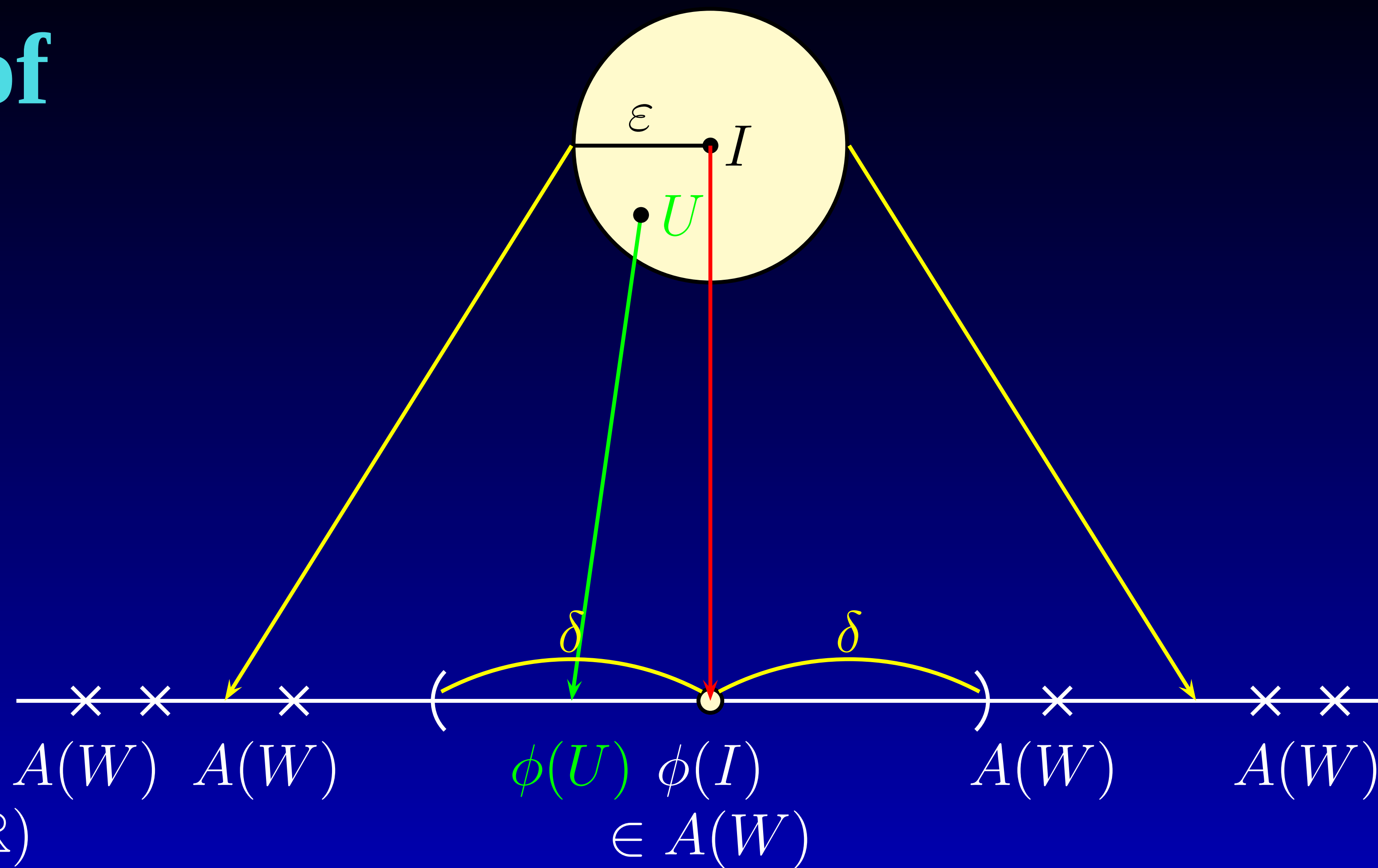
$\exists U \in SO(m, \mathbb{R})$

$\|U - I\| < \varepsilon$

$A(W) \not\ni \phi(U)$

Proof





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$\|U - I\| < \varepsilon \quad \Longrightarrow \quad W(U)$ is ε-close to W

$A(W) \not\ni \phi(U) \quad \in A(W(U))$

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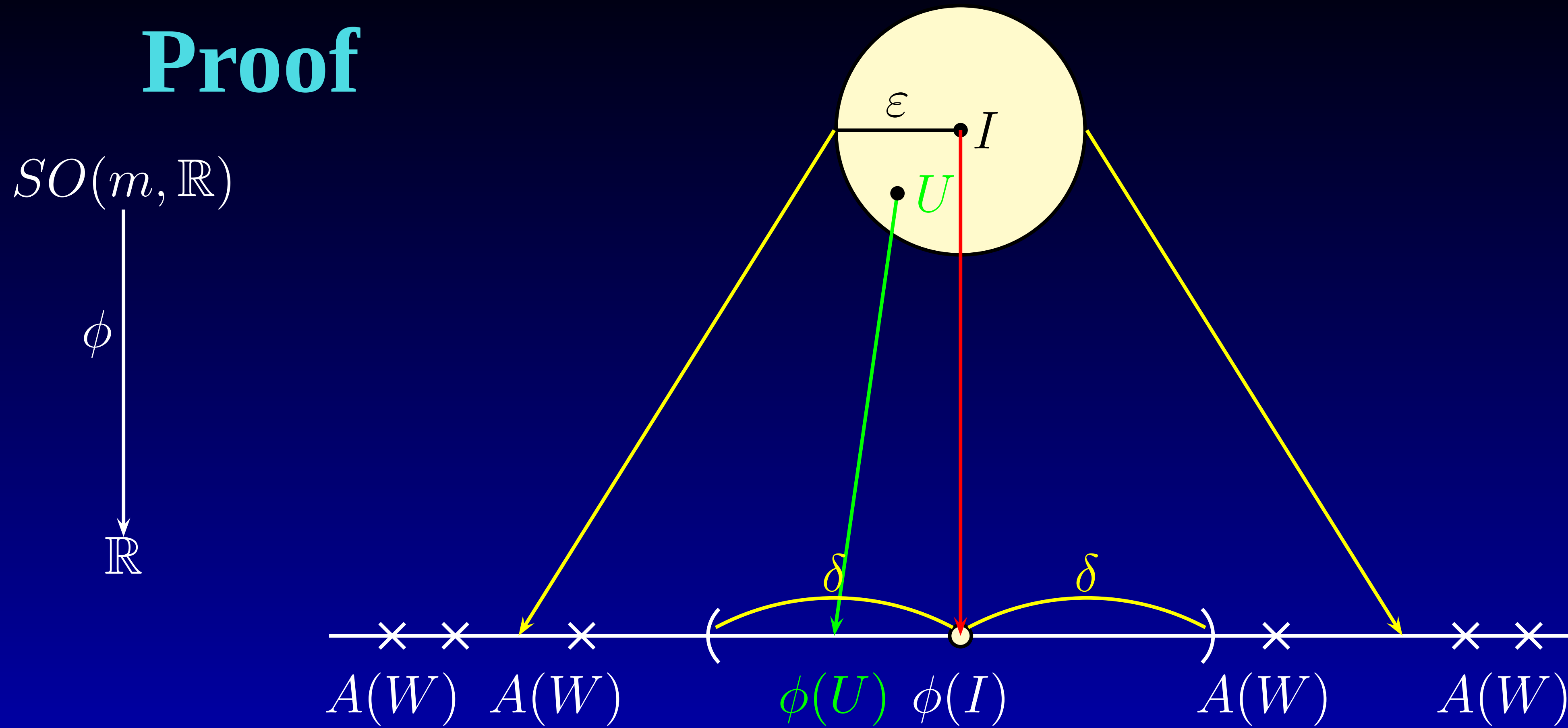

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so $A(W) \neq A(W(U))$, and hence $W \not\cong W(U)$.

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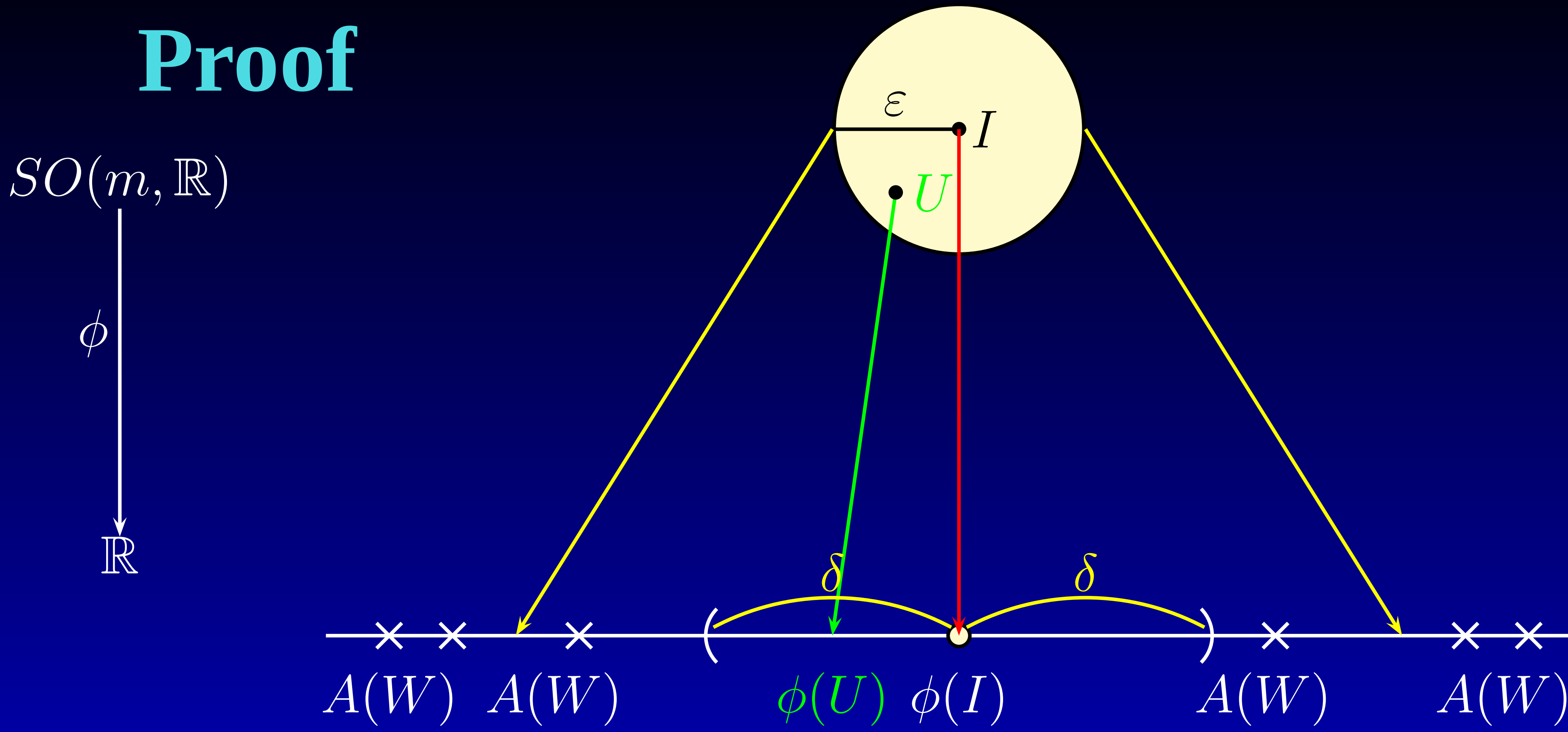

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THE END.