

Codes and Lattices of Hadamard Matrices

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June 12, 2007.

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Normalized if the entries of the first row are all 1.

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What if $p = 2$?

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Theorem

Let H be a Hadamard matrix of order 24 whose first row is the all-ones vector. The following statements are equivalent:

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Proof uses “Neighbors” of $\Lambda(H)$.

Neighbors

$$\frac{1}{2\sqrt{2}}\mathbb{Z}^{49} \begin{bmatrix} H + J \\ 8I \\ 4e_1 + \mathbf{1} \end{bmatrix} \\ = \Lambda(H)$$

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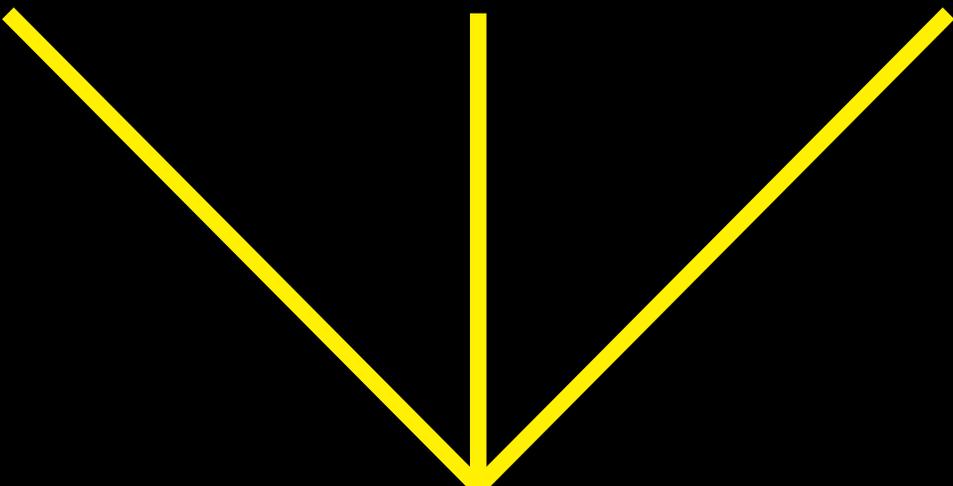
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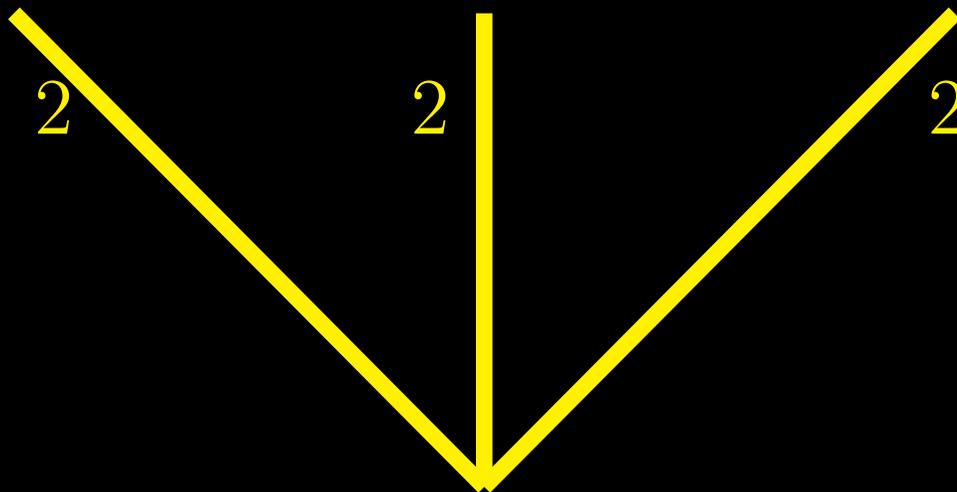
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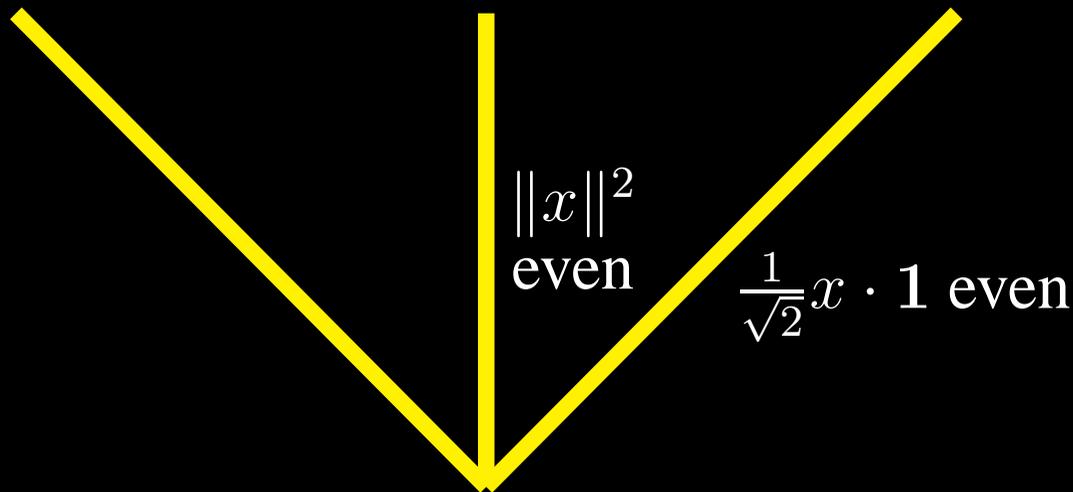
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$$\min \Lambda(H) = \min \Lambda_0(H)$$

$$\|x\|^2 \text{ even}$$

$$\frac{1}{\sqrt{2}}x \cdot \mathbf{1} \text{ even}$$

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