On Graphs with Complete Multipartite μ-Graphs

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(joint work with A. Jurišić and Y. Tagami)

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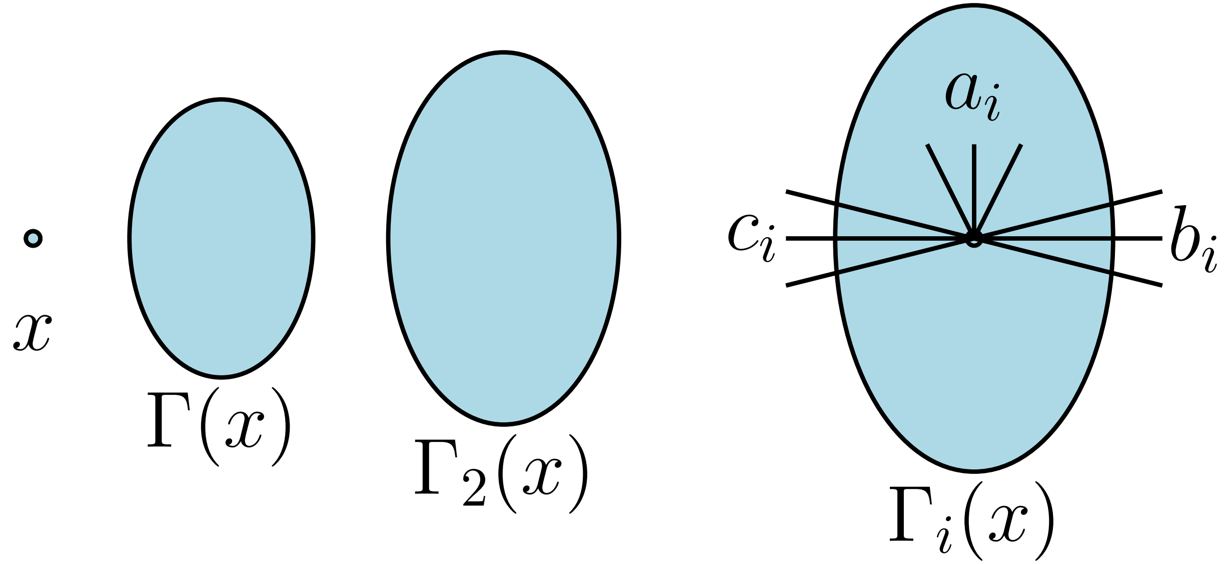
Bled, June 25, 2007.

Diagram



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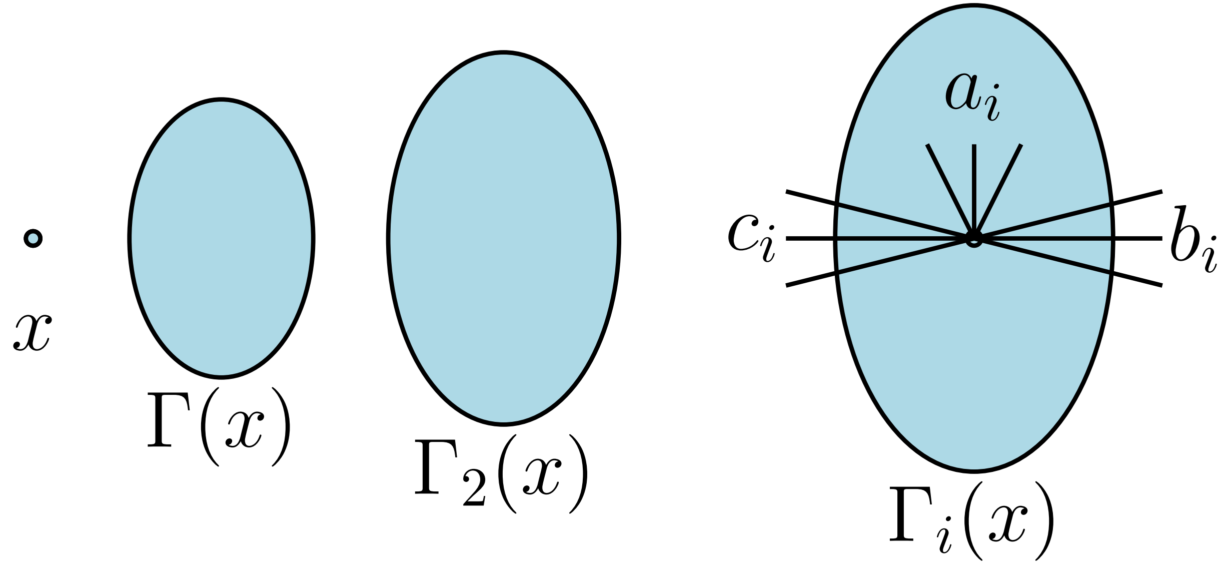
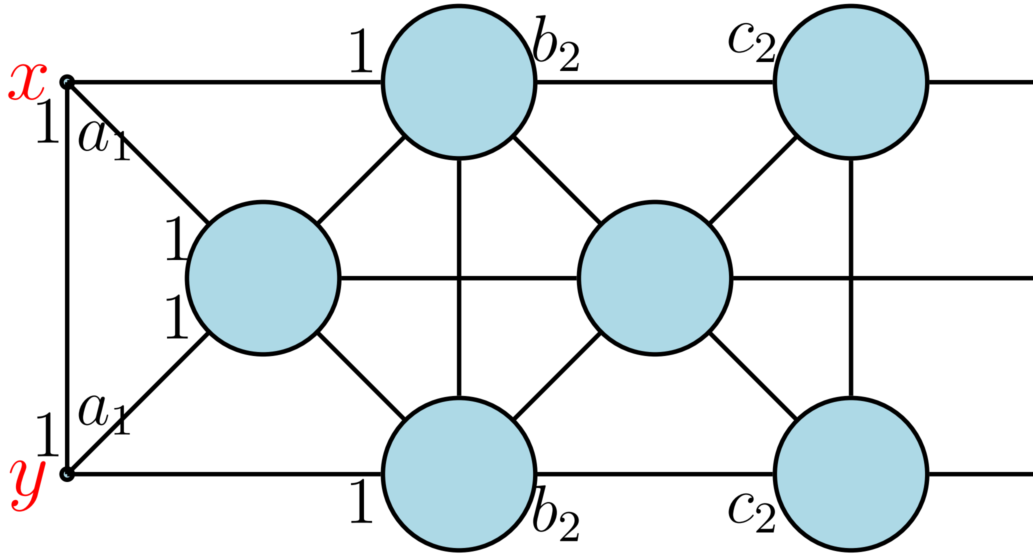


Nomura (1987)



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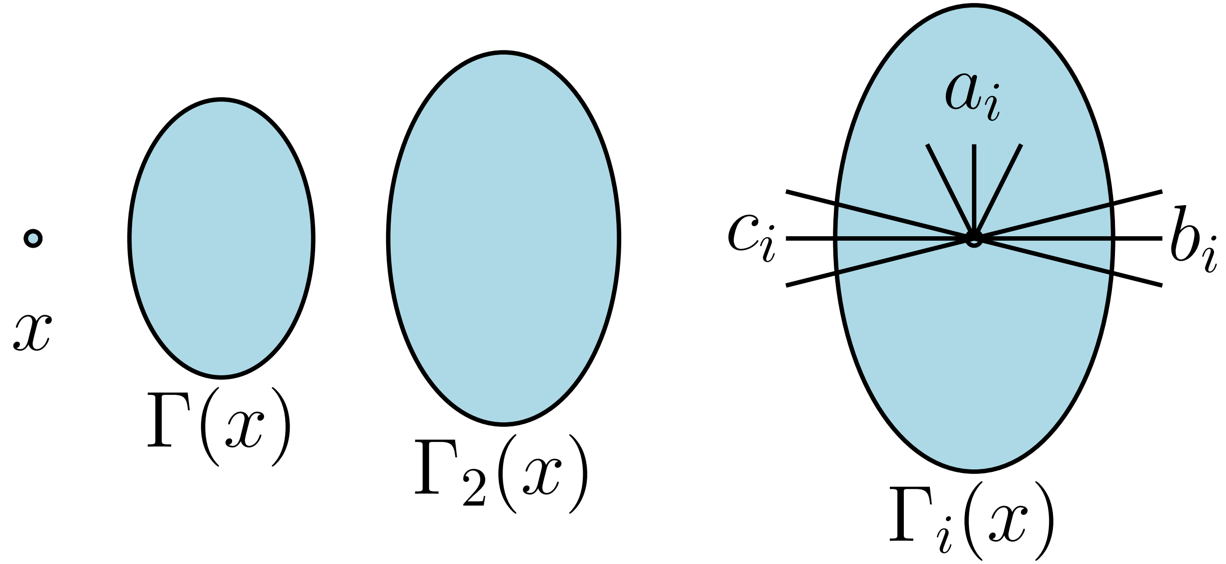

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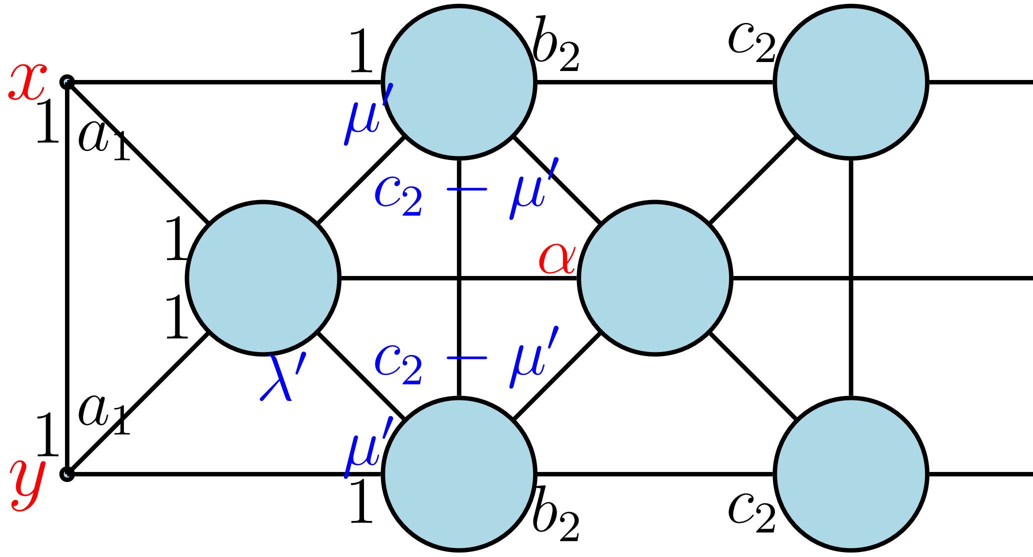

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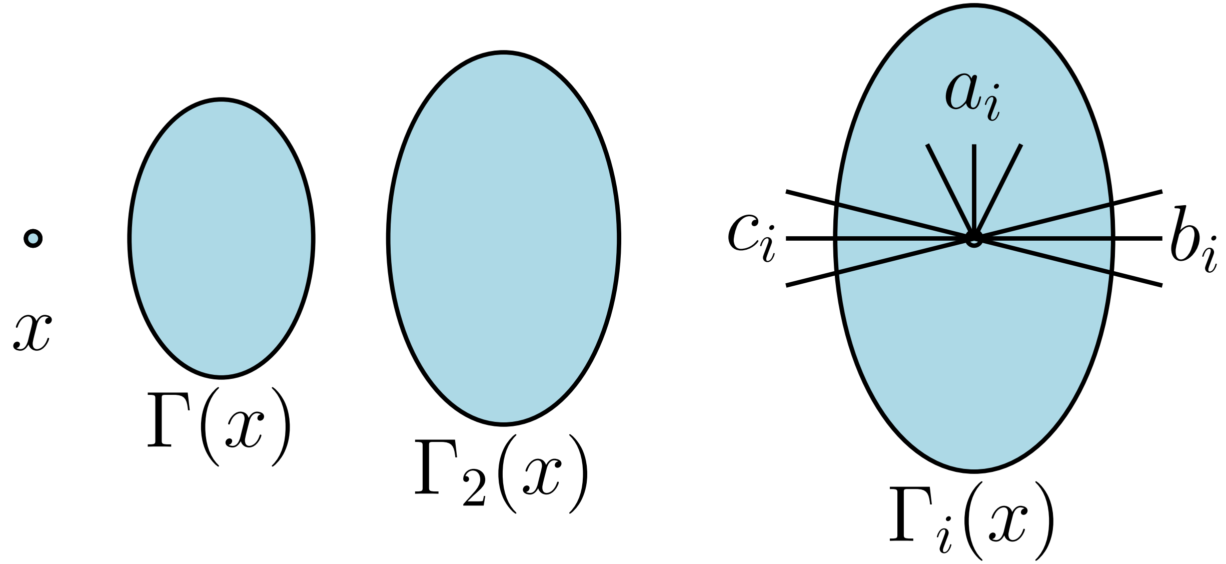

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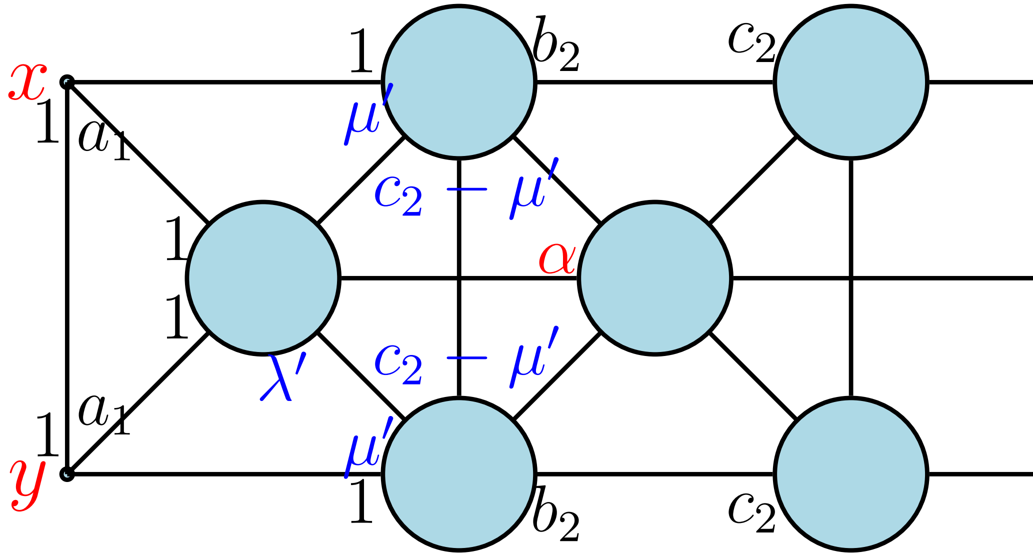

1-homogeneous

Jurišić–Koolen–Terwilliger (2000)

Local Graph

Γ : 1-homogeneous DRG

$\implies \Gamma(x)$: SRG with parameters $(v' = k, k' = \lambda, \lambda', \mu')$.

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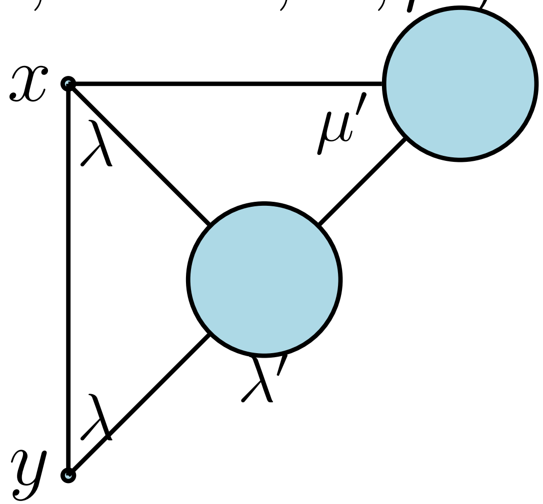

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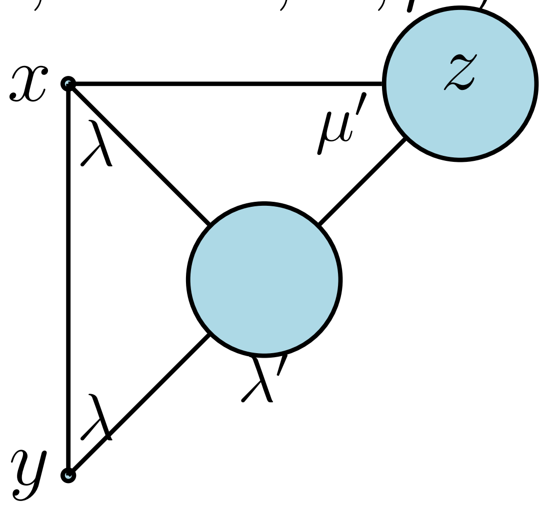
μ' is the valency of the μ-graph
(of size $\mu = |\Gamma(y) \cap \Gamma(z)|$, $d(y, z) = 2$).



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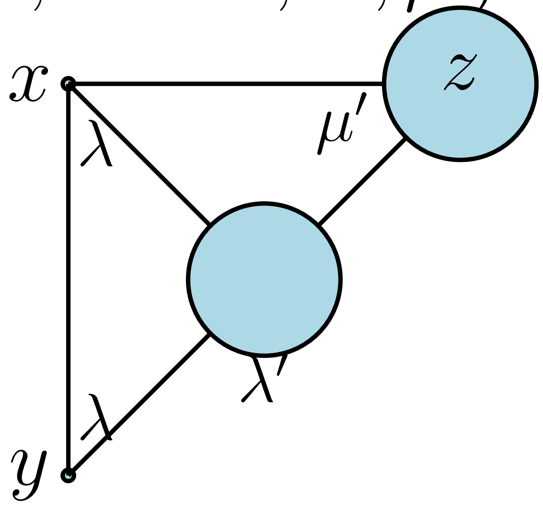

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Equality $\iff$ μ-graph $\cong K_\mu$.

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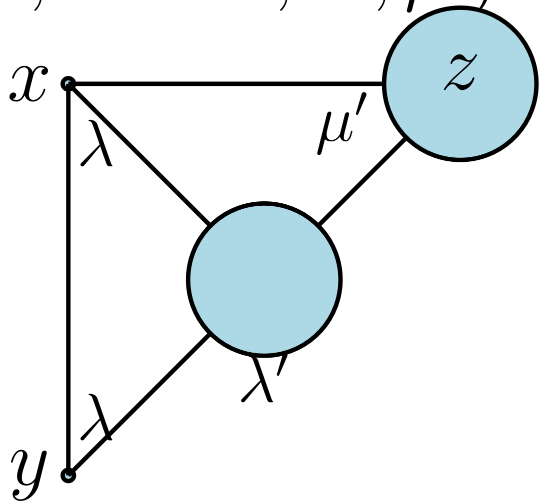

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Jurišić–Koolen (2000) classified 1-homogeneous DRG with $\mu' = \mu - 1$.

Only three such graphs.

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More generally,

$$\mu\text{-graph} \cong K_{t \times n}\text{: complete multipartite graph}$$

$K_{t \times n}$: t parts of $\overline{K_n}$

cocktail party graph = $K_{t \times 2}$

Examples

Antipodal DRG of diameter 4 (Jurišić–Koolen, 2007)

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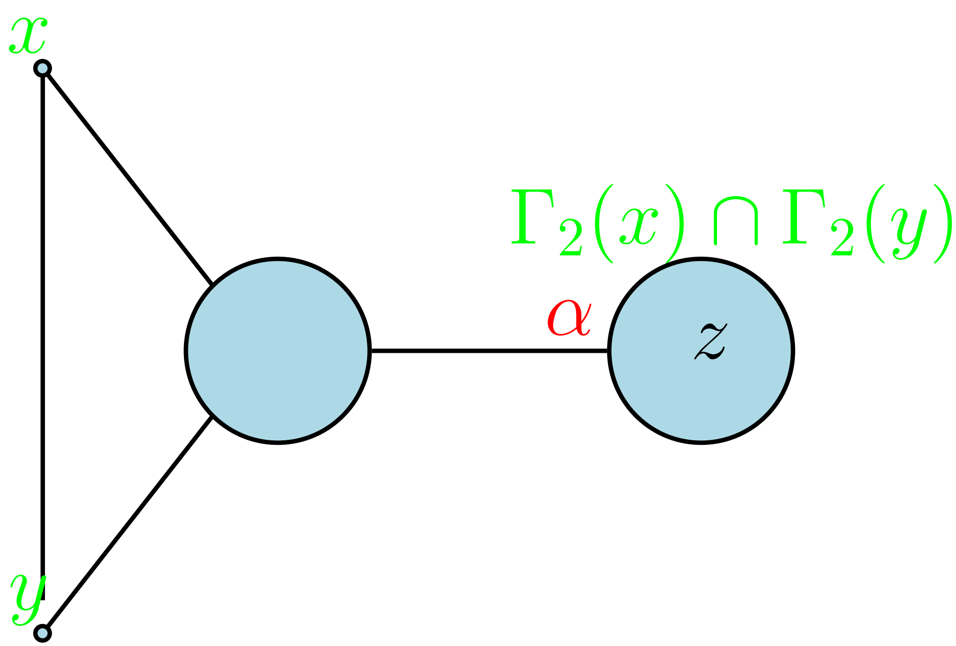
n	$t=2$	$t=3$	$t=4$
2	$J(8,4)$		
2	$J(8,2)$	$\frac{1}{2}H(8,2)$	
3	$NO_6^-(2)$	$NO_3^+(3)$	$3.O_7(3)$
4	$NU(5,2)$	Meixner2	
4	Patterson		

The Intersection Number α



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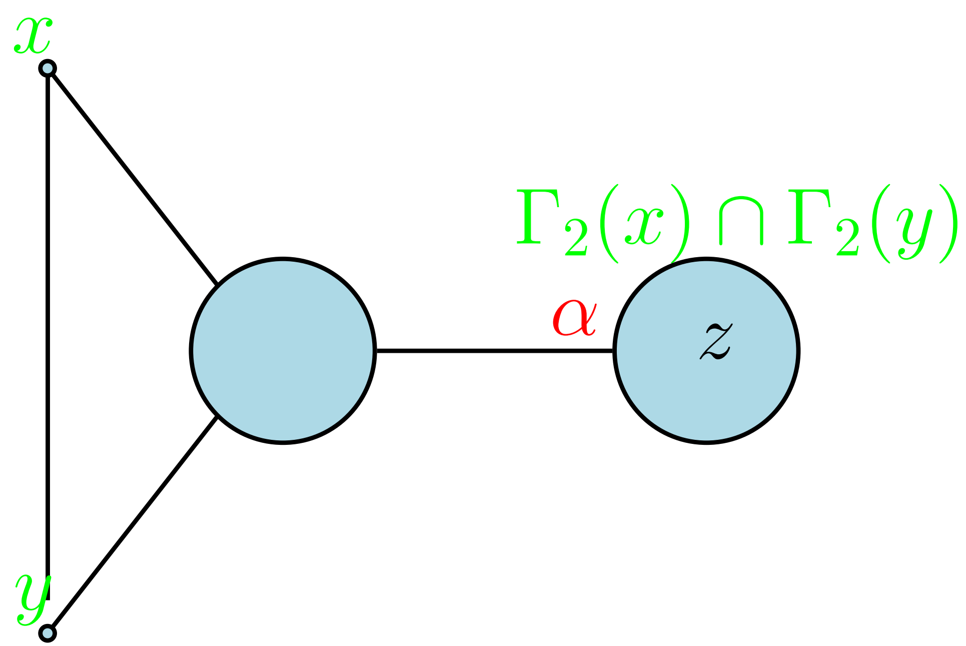

If $|\Gamma(x, y, z)|$ is constant whenever $d(x, y) = 1$, $z \in \Gamma_2(x) \cap \Gamma_2(y)$ (and there exists at least one such triple x, y, z), then we say

α exists

and denote this constant as α.

Conjecture of Jurišić–Koolen

Proposition (Jurišić–Koolen, 2003). Γ : DRG, locally SRG, μ-graph $\cong K_{t\times n}$, α exists $\implies \alpha \in \{t, t-1\}$.

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Consider the special case: $d = 2$, $t = 2$, $\alpha = 1$. Then Γ is a SRG with

$$k = 2n(2n^2 - 3n + 2), \quad \lambda = 2n^2 - 2n + 1, \quad \mu = 2n$$

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Conjecture (Jurišić–Koolen, 2007). $\nexists$ DRG, $d \geq 2$, locally SRG, α exists $\alpha = t - 1$, μ-graph $\cong K_{t\times n}$.

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- $t \leq 4$.
- $t = 4 \iff n = 3$.

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$\Leftarrow$: “local graph”

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Examples

			$\alpha = 2$		$\alpha = 3$		$\alpha = 4$
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2	2×6	$\Leftarrow$	$J(8,2)$	$\Leftarrow$	$\frac{1}{2}H(8,2)$		
3	$GQ(2,2)$	$\Leftarrow$	$NO^-_6(2)$	$\Leftarrow$	$NO^+_3(3)$	$\Leftarrow$	$3.O_7(3)$
4	$GQ(3,3)$	$\Leftarrow$	$NU(5,2)$	$\Leftarrow$	Meixner2		
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SRG		DRG
$n, t-1, \alpha-1$	$\Leftarrow$	n, t, α

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SRG $n, 2, 1$ ← if $\alpha = t - 1$ — SRG $n, t-1, \alpha-1$ $\Leftarrow$ DRG n, t, α

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- $t \leq 4$, with equality holds only if $n = 3$.