Spherical Designs Obtained from Certain Non-Unimodular Integral Lattices

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Delsarte

Delsarte (1973): X: Q-polynomial association scheme.

$$\left.\begin{array}{l} \text{Y: } t\text{-design} \\ s\text{-distance set} \\ t \geq 2s-2 \end{array}\right\} \implies Y : \text{Q-polynomial association scheme.}$$

Example: $S(5, 8, 24)$, $t = 5$. $s = 3$: $|B \cap B'| \in \{0, 2, 4\}$.
Delsarte–Goethals–Seidel (1977):

$$\left.\begin{array}{l} Y\text{: spherical } t\text{-design} \\ s\text{-distance set} \\ t \geq 2s-2 \end{array}\right\} \implies Y : \text{Q-polynomial assoc. scheme.}$$

Spherical t-design: $Y \subset S^{n-1} \subset \mathbb{R}^n$,

$$\frac{\int_{S^{n-1}} f \, d\mu}{\int_{S^{n-1}} 1 \, d\mu} = \frac{1}{|Y|} \sum_{x \in Y} f(x) \quad \text{whenever } \deg f \leq t$$

Etsuko Bannai 2007

s-Distance Set: $Y \subset S^{n-1} \subset \mathbb{R}^n$,

$$s = \#\{\|x - y\|^2 \mid x, y \in Y,\ x \neq y\}.$$

$$\left.\begin{array}{l} Y\text{: spherical } t\text{-design} \\ s\text{-distance set} \\ t \geq 2s - 3 \\ Y\text{: antipodal} \end{array}\right\} \Longrightarrow Y\text{ : Q-polynomial assoc. scheme.}$$

Example: E_6 root system Y. $s = |\{1, 0, -1, -2\}| = 4$, $t = 5$.

~~Example~~: Y: shortest (norm 6) vectors of Martinet's 10-dimensional lattice. $|Y| = 240$.

This is a lattice with the smallest dimension except root lattices whose set of vectors of a given norm carries a spherical 5-design (Nebe–Venkov 2000).

PNU in 2004

Talk given at Pusan National University by A. M.:

Y: shortest (norm 6) vectors of Martinet's 10-dimensional lattice. $|Y| = 240$.

$s = |\{3, 2, 1, 0, -1, -2, -3, -6\}| = 8$, $t = 5$.

Y carries an association scheme which is not Q-polynomial

Sp(4, 3)

Sp(4, 3) $\curvearrowright$ 80 points: multiplicity-free

5-dim. irreducible representation over $\mathbb{Q}(\sqrt{-3})$.

$\implies Z \subset \mathbb{R}^{10}$, $|Z| = 80$, consisting of elements of norm 3,

Z: 4-distance set ($4 = |\{1, 0, -1, -3\}|$)

but not 5-design (only 3-design).

$$Y = \langle\omega\rangle Z \subset \mathbb{Q}(\sqrt{-3})^5 \cong \mathbb{R}^{10}$$

is a 5-design, where $\omega = (-1 + \sqrt{-3})/2$.

Dimension 28

R. Bacher and B. Venkov (2001) classified 28-dim. unimodular lattices with min.= 3.

There are exactly 38 such lattices.

Harada–M.–Venkov (2007) classified ternary self-dual $[28, 14, 9]$ codes. There are 6,931 such codes.

One of the 38 lattices come from $\mathrm{Sp}(6, 3)$, due to Bacher–Venkov (1995).

Sp(6, 3)

$\mathrm{Sp}(6,3) \curvearrowright 2240$ points: multiplicity-free

14-dim. irreducible representation over $\mathbb{Q}(\sqrt{-3})$.

$\implies Z \subset \mathbb{R}^{28}$, $|Z| = 2240$, consisting of elements of norm 3,

Z: 4-distance set $(4 = |\{1, 0, -1, -3\}|)$

but not 5-design (only 3-design).

$\Lambda = \langle Z \rangle$ is one of the 38 unimodular lattices with min.= 3.

Harada–M.–Venkov

ternary self-dual [28, 14, 9] codes C with $A_3(C) = \Lambda = \langle Z \rangle$

$\iff$

sets of 28 pairwise orthogonal lines among the 1120 lines $= Z/\{\pm 1\}$, up to automorphism

$\iff$

symplectic spread (classified by Dempwolff (1994)).

$\mathrm{Sp}(6, 3) \curvearrowright$

$$\begin{aligned} 1120 \text{ lines} &= \text{max. totally isotropic subsp.} \\ &= \text{dual polar space } C_3(3) \end{aligned}$$

5-Design with 6720 points

	# lines	kissing #	$\times\langle\omega\rangle$
Sp(4, 3)	40	80	240
Sp(6, 3)	1120	2240	6720

Theorem. There exists a spherical 5-design of 6720 points in $\mathbb{R}^{28}$.

$\mathrm{Sp}(6,3) \curvearrowright$ 2240 points: multiplicity-free
14-dim. irreducible representation over $\mathbb{Q}(\sqrt{-3})$.
$\implies Z \subset \mathbb{R}^{28}$, $|Z| = 2240$, consisting of elements of norm 3,
$Y = \langle\omega\rangle Z$. Also,

$$Y \cong \{v \in \Lambda_0^* \mid (v, v) = 3\}. \quad \text{(dual of the even sublattice)}$$

2240 vectors of Λ, and 2×2240 vectors of the shadow of Λ.