On some cyclotomic association schemes and strongly regular graphs

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(joint work with Takuya Ikuta)

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Association Schemes

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$$
\begin{array}{cccccc}
 & A_0 & A_1 & \cdots & A_d & \\
 & I & \begin{bmatrix} \ddots & \\ & \ddots \end{bmatrix} & \cdots & \begin{bmatrix} \ddots & \\ & \ddots \end{bmatrix} & \text{diagonalize} \\
 & \downarrow & \downarrow & & \downarrow & \\
\text{mult.} = 1 & 1 & k_1 & \cdots & k_d & \\
\text{mult.} = m_1 & 1 & p_{11} & \cdots & p_{1d} & \\
\vdots & \vdots & \vdots & & \vdots & \\
\text{mult.} = m_d & 1 & p_{d1} & \cdots & p_{dd} &
\end{array}
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P_0 is called the principal part.

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complementary pair of strongly regular graphs (DRG with diameter= 2).

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Consider, more generally, $\mathrm{GF}(q)^\times = \langle\alpha\rangle \supset \langle\alpha^e\rangle$.

Cyclotomic Schemes

$$\mathrm{GF}(q)^\times = \langle \alpha \rangle \supset \langle \alpha^e \rangle, \quad e | q-1, \quad S \subset \mathbb{Z}_e = \{0, 1, \dots, e-1\}.$$

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$\Gamma_S = \mathrm{Cay}\left(\mathrm{GF}(q), \bigcup_{i\in S} \alpha^i\langle\alpha^e\rangle\right)$.

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	A_0	A_1	$\cdots$	A_e
	$\updownarrow$	$\updownarrow$	$\cdots$	$\updownarrow$
	$\emptyset$	$\Gamma_{\{0\}}$	$\cdots$	$\Gamma_{\{e-1\}}$
mult.$= 1$	1	f	$\cdots$	f
mult.$= f$	1			
$\vdots$	$\vdots$	P_0: circulant $e \times e$		
mult.$= f$	1			

Uniform Cyclotomy

Recall Clebsch: $q = 16$, $e = 3$.

$$\begin{bmatrix} 1 & 5 & 10 \\ 1 & -3 & 2 \\ 1 & 1 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 5 & 5 & 5 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{bmatrix}$$

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Bannai (1991), Muzychuk (1987): subscheme, fusion scheme, fusing relations.

$$P = \begin{bmatrix} 1 & f & \cdots & f \\ 1 & r & \cdots & s \\ \vdots & \vdots & \ddots & \\ 1 & s & \cdots & r \end{bmatrix}$$

Uniform Cyclotomy

In general, if a cyclotomic scheme has

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$\iff q = p^{2m}, e | p^{m'} + 1, m' | m$. Baumert–Mills–Ward (1982), Brouwer–Wilson–Xiang (1999).

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An association scheme $(A_0 = I, A_1, \ldots, A_d)$ is amorphous if

$$\forall \emptyset \neq S \subset \{1, \ldots, d\}, \sum_{i \in S} A_i \text{ is SRG}.$$

van Dam–Muzychuk (preprint)

GF(2^{12})

$q = 2^{12}$, $e = 45$, $\Gamma_{\{0\}} = \mathrm{Cay}(\mathrm{GF}(q), \langle \alpha^e \rangle)$: not SRG,
eigenvalues 91, 19, 11, 3, −5, −13

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$\Gamma_{\{0,5,10\}}$: SRG (de Lange 1995), eigenvalues 273, 17, -15.

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$$\begin{bmatrix} 1 & 273 & 273 & 273 & 3276 \\ 1 & 17 & 17 & 17 & -52 \\ 1 & 17 & -15 & -15 & 12 \\ 1 & -15 & 17 & -15 & 12 \\ 1 & -15 & -15 & 17 & 12 \end{bmatrix}$$

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$\Gamma_{\{0,5,10\}+3i}$ $(i = 0, \ldots, 14)$. P_0 : $(17, -15)$ inc. mat. PG$(3, 2)$

Symmetric Designs

$$
\begin{array}{rc|ccc|}
 & A_0 & A_1 & \cdots & A_d \\
\text{mult.}= 1 & 1 & f & \cdots & f \\
\hline
\text{mult.}= f & 1 & & & \\
\vdots & \vdots & & P_0 & \\
\text{mult.}= f & 1 & & (d \times d) & \\
\hline
\end{array}
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(pseudocyclic scheme)

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(pseudocyclic scheme)

Theorem. If P_0 has 2 distinct entries r and s, then P_0 is an (r, s)-incidence matrix of a symmetric design (possibly $P_0 \in \langle I, J \rangle$).

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Example.

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Example.

- $q = 2^{12}, e = 45 \rightarrow d = 15, P_0 : \mathrm{PG}(3, 2)$
- $q = 2^{20}, e = 75 \rightarrow d = 15, P_0 : \mathrm{PG}(3, 2)$

Symmetric Designs

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Proof is immediate from the orthogonality relations.

Example.

- $q = 2^{12}, e = 45 \rightarrow d = 15$, P_0 : PG(3, 2)
- $q = 2^{20}, e = 75 \rightarrow d = 15$, P_0 : PG(3, 2)
- $q = 2^{21}, e = 49 \rightarrow d = 7$, P_0 : PG(2, 2)

PG(m, q)

$$
\begin{array}{r|c c c|}
 & A_0 & A_1 \cdots A_{q+1} \quad \cdots & A_d \\
\hline
\text{mult.}= 1 & 1 & f \cdots f \quad \cdots & f \\
\text{mult.}= f & 1 & & \\
\vdots & \vdots & P_0 : \quad \text{PG}(m, q) & \\
\text{mult.}= f & 1 & d \times d & \\
\hline
\end{array}
$$

$$d = \frac{q^{m+1} - 1}{q - 1}.$$

PG(m, q)

	A_0	$A_1 \cdots A_{q+1}$	$\cdots$	A_d
mult.= 1	1	$f \cdots f$	$\cdots$	f
mult.= f	1			
$\vdots$	$\vdots$	P_0 :	PG(m, q)	
mult.= f	1		$d \times d$	

$$d = \frac{q^{m+1} - 1}{q - 1}.$$

$A_1, \dots, A_{q+1}$: points on a line of PG(m, q).

$PG(m, q)$

	A_0	$A_1 \cdots A_{q+1}$	$\cdots$	A_d
mult.= 1	1	$f \cdots f$	$\cdots$	f
mult.= f	1			
$\vdots$	$\vdots$	P_0 :	$PG(m, q)$	
mult.= f	1		$d \times d$	

$$d = \frac{q^{m+1} - 1}{q - 1}.$$

$A_1, \ldots, A_{q+1}$: points on a line of $PG(m, q)$.

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1	$f \cdots f$	$(d - q - 1)f$
1		
$\vdots$	$(q + 2) \times (q + 2)$	
1		

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2^{12} (van Dam 2003)

$$\begin{bmatrix} 1 & 273 & 273 & 273 & 3276 \\ 1 & 17 & 17 & 17 & -52 \\ 1 & 17 & -15 & -15 & 12 \\ 1 & -15 & 17 & -15 & 12 \\ 1 & -15 & -15 & 17 & 12 \end{bmatrix}$$

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Spreads in $\mathrm{PG}(3, q)$

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$\exists$ another spread, (although spreads are equivalent under $\mathrm{GL}(4, 2)$) the resulting SRG (hence amorphous scheme) is not isomorphic to the above cyclotomic example.

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$GF(3^{12})$	$(35, 17, 8)$	$15184,\ 118,\ -125$
$GF(5^9)$	$(19, 9, 4)$	$1953125,\ 296,\ -329$
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