Steiner quadruple systems extending affine triple systems

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Köhler (Fitting?) graph

Fitting (1915). $A = \mathbb{Z}_{34}$

Fitting (1915), Key–Wagner (1986)

Construction of Steiner systems

(1) Fitting considered a graph associated to a cyclic group of order v in order to construct a cyclic 3-$(v, 4, 1)$ design.

(2) Key and Wagner noticed ovals can be used to extend AG(d, q) to a 3-$(q^d + 1, q + 1, 1)$ design.

(1)' Consider a graph associated to an abelian group of order v in order to construct a 3-$(v, 4, 1)$ design.

(2)' Ovals can be used to extend AG$(d, 3)$ to a 3-$(3^d + 1, 4, 1)$ design.

However,

(1)" An abelian group acts regularly on points.

(2)" The abelian group $\mathbb{Z}_3^d$ fixes the extended point and acts regularly on the rest.

Regular action of abelian group

Construction of Steiner systems

- $A =$ a finite abelian group, $\hat{A} = A \rtimes \langle \tau \rangle$, $a^\tau = -a$ $(a \in A)$.
- $\hat{A}$ acts on A, and also on $\binom{A}{3}$ and on $\binom{A}{4}$.
- Köhler graph of $A = (\mathcal{T}, \mathcal{Q})$, where

$$\mathcal{T} \subset \binom{A}{3} / \hat{A}, \quad \mathcal{Q} \subset \binom{A}{4} / \hat{A}$$

  are "generic" triples and quadruples.
- Generic means, for example, for $A = \mathbb{Z}_3^d$, non-collinear points in AG$(d, 3)$.

Example

$A =$ a finite abelian group, $\hat{A} = A \rtimes \langle \tau \rangle$, $a^{\tau} = -a$ $(a \in A)$.

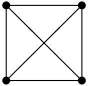
- $A = \mathbb{Z}_3^2$
- $\mathcal{T}$ consists of $\hat{A}$-orbits of non-collinear triples of $A = \mathrm{AG}(2,3)$.

ovals (6 orbits)

non-collinear triples

$\{(0,0),(1,0),(0,1)\}$
$\{(0,0),(1,0),(0,2)\}$
$\{(0,0),(1,0),(2,2)\}$
$\{(0,0),(0,1),(2,2)\}$

incidence matrix of graph



Kramer–Mesner matrix

$A = \mathbb{Z}_3^2$

The $\binom{A}{3} \times \binom{A}{4}$ Kramer–Mesner matrix.

$\binom{A\cup\{\infty\}}{4}$

		2 oval orbits	$\infty\cup$line	$\cdots$
$\binom{A\cup\{\infty\}}{3}$	$\infty\cup$ 2 points	0	1	?
	lines	0	1	?
	non-collinear	1 1	0 0	? ?

Aim:

- Generalize this construction to $A = \mathbb{Z}_3^d$ for $d \geq 3$, and give a lower bound on the number of isomorphism classes of SQS$(3^d + 1)$ which extend AG$(d, 3)$.
- Describe analogy with Fitting's method.