Twisted Grassmann graph is the block graph of a design

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Designs and Codes, June 21–24, 2009

Notation

- e: a positive integer,
- V: a $(2e+1)$-dimensional vector space over GF(q).
- for a subset W of V closed under multiplication by the elements of GF(q), denote by $[W]$ the set of 1-dimensional subspaces (projective points) contained in W.
- for a vector space W, denote by $\begin{bmatrix} W \\ k \end{bmatrix}$ the set of k-dimensional subspaces of W,

The geometric design $\mathrm{PG}_e(2e,q)$ has $[V]$ as the set of points, and $\{[W] \mid W \in \begin{bmatrix} V \\ e+1 \end{bmatrix}\}$ as the set of blocks. The block graph of this design, where two blocks $[W_1],[W_2]$ are adjacent whenever $\dim W_1 \cap W_2 = e$, is the Grassmann graph $J_q(2e+1,e+1)$ which is isomorphic to the Grassmann graph $J_q(2e+1,e)$.

Twisted Grassmann graph

Let H be a fixed hyperplane of V. The twisted Grassmann graph has the set of vertices $\mathcal{A} \cup \mathcal{B}$, where

$$\mathcal{A} = \{W \in \begin{bmatrix} V \\ e+1 \end{bmatrix} \mid W \not\subset H\},$$
$$\mathcal{B} = \begin{bmatrix} H \\ e-1 \end{bmatrix}.$$

The adjacency is defined as follows:

$$W_1 \sim W_2 \iff \begin{cases} \dim W_1 \cap W_2 = e & \text{if } W_1 \in \mathcal{A},\ W_2 \in \mathcal{A}, \\ W_1 \supset W_2 & \text{if } W_1 \in \mathcal{A},\ W_2 \in \mathcal{B}, \\ \dim W_1 \cap W_2 = e-2 & \text{if } W_1 \in \mathcal{B},\ W_2 \in \mathcal{B}. \end{cases}$$

The design constructed by Jungnickel–Tonchev

Let σ be a polarity of H.
The pseudo-geometric design constructed by Jungnickel and Tonchev has $[V]$ as the set of points, and $\mathcal{A}' \cup \mathcal{B}'$ as the set of blocks, where

$$\mathcal{A}' = \{[\sigma(W \cap H) \cup (W \setminus H)] \mid W \in \mathcal{A}\},$$
$$\mathcal{B}' = \{[W] \mid W \in \begin{bmatrix} H \\ e+1 \end{bmatrix}\}.$$

The incidence structure $([V], \mathcal{A}' \cup \mathcal{B}')$ is a 2-(v, k, λ) design, where

$$v = \frac{q^{2e+1}-1}{q-1},\ k = \frac{q^{e+1}-1}{q-1},\ \lambda = \frac{(q^{2e-1}-1)\cdots(q^{e+1}-1)}{(q^{e-1}-1)\cdots(q-1)}.$$

The isomorphism

Theorem

The twisted Grassmann graph is isomorphic to the block graph of the design $([V], \mathcal{A}' \cup \mathcal{B}')$, where two blocks are adjacent if and only if their intersection has size $(q^e - 1)/(q - 1)$.

Proof. We define a mapping $f : \mathcal{A} \cup \mathcal{B} \to \mathcal{A}' \cup \mathcal{B}'$ by

$$f(W) = \begin{cases} [\sigma(W \cap H) \cup (W \setminus H)] & \text{if } W \in \mathcal{A}, \\ [\sigma(W)] & \text{if } W \in \mathcal{B}. \end{cases}$$

Then we show:

$$W_1 \sim W_2 \iff |f(W_1) \cap f(W_2)| = \frac{q^e - 1}{q - 1}$$

References

- E.R. van Dam and J.H. Koolen, A new family of distance-regular graphs with unbounded diameter, Invent. Math. 162 (2005), 189-193.
- D. Jungnickel and V.D. Tonchev, Polarities, quasi-symmetric designs, and Hamadafs conjecture, Des. Codes Cryptogr. 51 (2009), 131–140.