A characterization of quasi-line graphs

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Graphs and Line Graphs

A graph $G = (V, E)$ consists of a finite set V together with a set E of two-element subsets of V.

- V: vertices
- E: edges

The line graph $L(G)$ of a graph G has E as the set of vertices and its set of edges is

$$\{\{e, f\} \mid e, f \in E,\ |e \cap f| = 1\}.$$

Properties of Line Graphs

G: a graph
$\Gamma = L(G)$: the line graph of G.

(i) The neighborhood of every vertex in Γ is a union of two subsets, each of which is a clique.

(ii) The graph Γ admits a representation by vectors of squared norm 2 in $\mathbf{Z}^n$, where $n = |V|$.

Here, a representation by vectors of squared norm 2 means a mapping $\phi : V(\Gamma) \to \mathbf{Z}^n$ such that $\|\phi(e)\|^2 = 2$ for $e \in E(\Gamma)$, $(\phi(e), \phi(f)) = 1$ or 0, according as $\{e, f\} \in E(\Gamma)$ or not.
A graph satisfying (i) is called a quasi-line graph, while a graph satisfying (ii) is called a generalized line graph. (ii) means the image of ϕ is contained in the root system of type D.

Representation of Graphs by Vectors

The incidence matrix:

$$\text{edges} \;=\; \text{vertices of } L(G) = \begin{array}{c} (1\;1\;0\;0\;0) \\ (0\;1\;1\;0\;0) \\ (1\;0\;0\;1\;0) \\ (0\;0\;1\;1\;0) \\ (0\;0\;1\;0\;1) \end{array}$$

Allowing -1

$$\begin{array}{cccccc} (& -1 & 1 & 0 & 0 & 0) \\ (& 0 & 1 & -1 & 0 & 0) \\ (& -1 & 0 & 0 & 1 & 0) \\ (& 0 & 0 & -1 & 1 & 0) \\ (& 0 & 0 & -1 & 0 & 1) \end{array} \subset \text{root system of type } A$$

Examples

A claw can be represented by the vectors of squared norm 2:

$$\begin{pmatrix}1 & 1 & 0 & 0\end{pmatrix}$$
$$\begin{pmatrix}1 & 0 & 1 & 0\end{pmatrix}$$
$$\begin{pmatrix}0 & 1 & 0 & 1\end{pmatrix}$$
$$\begin{pmatrix}1 & 0 & -1 & 0\end{pmatrix}$$

hence it is a generalized line graph, but it is not a quasi-line graph. The graph with adjacency matrix

$$\begin{bmatrix}0 & 1 & 1 & 0 & 0 & 0\\ 1 & 0 & 1 & 1 & 1 & 0\\ 1 & 1 & 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0 & 1 & 1\\ 0 & 1 & 1 & 1 & 0 & 1\\ 0 & 0 & 0 & 1 & 1 & 0\end{bmatrix}$$

is a quasi-line graph, but not a generalized line graph.

Exceptional Root Systems

If

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

then $A + 2I$ is a Gram matrix of the E_6-lattice, which is known not to be contained in D_n for any n. Hence the graph Γ with adjacency matrix A is not a generalized line graph. However, Γ is a quasi-line graph.

{line graph} ⊂ {generalized line graph}

{line graph} ⊂ {quasi-line graph} ⊂ {claw-free graph}

Quasi-line graph

Chudnovsky–Seymour gave a structural characterization of claw-free graphs, and also of quasi-line graphs. We wish to characterize quasi-line graphs using the concept of sums of Hoffman graphs.

Definition

A Hoffman graph is a graph H with vertex labeling $V(H) \to \{s, f\}$, satisfying the following conditions:

(i) every vertex with label f is adjacent to at least one vertex with label s;

(ii) vertices with label f are pairwise non-adjacent.

$s =$ "slim," $f =$ "fat."

$V_s(H)$ ($V_f(H)$) $=$ the set of slim (fat) vertices of H.

An ordinary graph $=$ Hoffman graph without fat vertex.

Sums of Hoffman graphs

Definition

Let H be a Hoffman graph, and let H^i $(i = 1, 2, \ldots, n)$ be a family of subgraphs of H. We write

$$H = \biguplus_{i=1}^{n} H^i$$

if

(i) $V(H) = \bigcup_{i=1}^{n} V(H^i)$;

(ii) $V_s(H^i) \cap V_s(H^j) = \emptyset$ if $i \neq j$;

(iii) if $x \in V_s(H^i)$ and $\alpha \in V_f(H)$ are adjacent, then $\alpha \in V(H^i)$;

(iv) if $x \in V_s(H^i)$, $y \in V_s(H^j)$ and $i \neq j$, then x and y have at most one common fat neighbour, and they have one if and only if they are adjacent.

$\hat{G}$ of a bipartite graph G

Given a bipartite graph G with bipartition $V_1 \cup V_2$,

$\hat{G} =$ Hoffman graph obtained from G by
making every pair of vertices in V_i adjacent,
attaching a common fat neighbor f_i to V_i,
for $i = 1, 2$

A construction of quasi-line graphs

Suppose

$$V(Q) = \bigcup_{i=1}^{n} V_i \quad \text{(disjoint cliques)}$$

$$V_i = \bigcup_{j=1}^{n} V_{ij}, \quad V_{ii} = \emptyset,$$

$$x \in V_{ij},\ y \notin V_i,\ x \sim y \implies y \in V_{ji}.$$

Then Q is a quasi-line graph which can be expressed as a sum of Hoffman graphs of the form $\hat{G}$.