Weighing matrices of order $2(q+1)$ and weight q

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Definition

A weighing matrix W of order n and weight k is an $n \times n$ matrix W with entries $1, -1, 0$ such that

$$WW^T = kI_n,$$

where I_n is the identity matrix of order n and W^T denotes the transpose of W.
We say that two weighing matrices W_1 and W_2 of order n and weight k are *equivalent* if there exist monomial matrices P and Q with $W_1 = PW_2Q$.

$n = k \implies$ Hadamard matrix

Chan–Roger–Seberry (1986)

Classified all weighing matrices of weight $k \leq 5$.

In particular, there are two weighing matrices of order 12 and weight 5, up to equivalence.

However, there is another one, discovered by Harada and A.M. recently, using the classification of self-dual codes of length 12 over $\mathbb{F}_5$.

Chan–Roger–Seberry (1986) missed:

+	0	+	0	0	−	0	0	0	−	+	0
0	+	+	+	+	+	0	0	0	0	0	0
+	+	0	0	−	0	0	0	−	+	0	0
0	+	0	0	0	−	−	0	+	0	−	0
0	+	−	0	0	0	+	0	0	−	0	+
−	+	0	−	0	0	0	0	0	0	+	−
0	0	0	−	+	0	−	0	−	0	0	+
0	0	0	0	0	0	0	−	−	−	−	−
0	0	−	+	0	0	−	−	0	0	+	0
−	0	+	0	−	0	0	−	0	0	0	+
+	0	0	−	0	+	0	−	+	0	0	0
0	0	0	0	+	−	+	−	0	+	0	0

Inner product on $\mathbb{F}_5^2$ over $\mathbb{F}_5$

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2.$$

	(0,1)	(1,0)	(1,1)	(1,2)	(1,3)	(1,4)
(0,1)	1	0	1	2	3	4
(1,0)	0	1	1	1	1	1
(1,1)	1	1	2	3	4	0
(1,2)	2	1	3	0	2	4
(1,3)	3	1	4	2	0	3
(1,4)	4	1	0	4	3	2

We change

$$1 \to +, \quad 4 \to -, \quad 2 \to 0, \quad 3 \to 0$$

Inner product on $\mathbb{F}_5^2$ over $\mathbb{F}_5$ $\rightarrow$ a 6×6 matrix with entries $0, \pm 1$

$A_1 =$

	(0,1)	(1,0)	(1,1)	(1,2)	(1,3)	(1,4)
(0,1)	$+$	0	$+$	0	0	$-$
(1,0)	0	$+$	$+$	$+$	$+$	$+$
(1,1)	$+$	$+$	0	0	$-$	0
(1,2)	0	$+$	0	0	0	$-$
(1,3)	0	$+$	$-$	0	0	0
(1,4)	$-$	$+$	0	$-$	0	0

Inner product on $\mathbb{F}_5^2$ over $\mathbb{F}_5$ $\longrightarrow$ a 6×6 matrix with entries $0, \pm 1$

	2(0,1)	2(1,0)	2(1,1)	2(1,2)	2(1,3)	2(1,4)
(0,1)	2	0	2	4	1	3
(1,0)	0	2	2	2	2	2
(1,1)	2	2	4	1	3	0
(1,2)	4	2	1	0	4	3
(1,3)	1	2	3	4	0	1
(1,4)	3	2	0	3	1	4

Replace

$$1 \longrightarrow +, \quad 4 \longrightarrow -, \quad 2 \longrightarrow 0, \quad 3 \longrightarrow 0$$

to obtain A_2.

Inner product on $\mathbb{F}_5^2$ over $\mathbb{F}_5$ $\to$ a 6×6 matrix with entries $0, \pm 1$

	(0,1)	(1,0)	(1,1)	(1,2)	(1,3)	(1,4)
2(0,1)	2	0	2	4	1	3
2(1,0)	0	2	2	2	2	2
2(1,1)	2	2	4	1	3	0
2(1,2)	4	2	1	0	4	3
2(1,3)	1	2	3	4	0	1
2(1,4)	3	2	0	3	1	4

Replace

$$1 \to +, \quad 4 \to -, \quad 2 \to 0, \quad 3 \to 0$$

to obtain A_3 (which is the same as A_2).

Inner product on $\mathbb{F}_5^2$ over $\mathbb{F}_5$ $\rightarrow$ a 6×6 matrix with entries $0, \pm 1$

	2(0,1)	2(1,0)	2(1,1)	2(1,2)	2(1,3)	2(1,4)
2(0,1)	4	0	4	3	2	1
2(1,0)	0	4	4	4	4	4
2(1,1)	4	4	3	2	1	0
2(1,2)	3	4	2	0	3	1
2(1,3)	2	4	1	3	0	2
2(1,4)	1	4	0	1	2	3

Replace

$$1 \rightarrow +, \quad 4 \rightarrow -, \quad 2 \rightarrow 0, \quad 3 \rightarrow 0$$

to obtain A_4 (which is the same as $-A_1$).

Notation

- q: a prime power, $q \equiv 1 \pmod 4$ (ex. $q = 5$)
- $F = GF(q)$: a finite field, $F^\times = \langle a \rangle$ (ex. $a = 2$)
- V: a vector space of dimension m over F, $m > 1$ (ex. $m = 2$)
- $V^\sharp = V \setminus \{0\}$
- $n = 2 \cdot (q^m - 1)/(q - 1)$ (ex. $n = 12$)
- $X = V^\sharp / \langle a^2 \rangle = \{\langle a^2 \rangle x_i \mid 1 \leq i \leq n\}$ ($|X| = n$)
- $B : V \times V \to F$: nondegenerate bilinear form (ex. $B(x, y) = x_1 y_1 + x_2 y_2$)

Define $n \times n$ matrix W by

$$W_{ij} = \begin{cases} 1 & \text{if } B(x_i, x_j) \in \langle a^4 \rangle, \text{ (ex. } \in \{1\}\text{)} \\ -1 & \text{if } B(x_i, x_j) \in a^2 \langle a^4 \rangle, \text{ (ex. } \in \{4\}\text{)} \\ 0 & \text{otherwise.} \end{cases}$$

Main result

- q: a prime power, $q \equiv 1 \pmod 4$
- $F = GF(q)$: a finite field, $F^\times = \langle a \rangle$
- V: a vector space of dimension m over F, $m > 1$
- $V^\sharp = V \setminus \{0\}$
- $n = 2 \cdot (q^m - 1)/(q - 1)$
- $X = V^\sharp / \langle a^2 \rangle = \{\langle a^2 \rangle x_i \mid 1 \le i \le n\}$ ($|X| = n$)
- $B : V \times V \to F$: nondegenerate bilinear form

$$W_{ij} = \begin{cases} 1 & \text{if } B(x_i, x_j) \in \langle a^4 \rangle, \\ -1 & \text{if } B(x_i, x_j) \in a^2 \langle a^4 \rangle, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem

W is a weighing matrix of order n and weight q^{m-1}.

$$W_{ij} = \pm 1 \iff B(x_i, x_j) \in \langle a^2 \rangle$$

$$\begin{aligned}
\text{weight} &= |\{j \mid 1 \le j \le n,\ B(x_i, x_j) \in \langle a^2 \rangle\}| \\
&= \sum_{j=1}^{n} \frac{1}{|\langle a^2 \rangle|} |\{y \in \langle a^2 \rangle x_j \mid B(x_i, y) \in \langle a^2 \rangle\}| \\
&= \frac{1}{|\langle a^2 \rangle|} \left| \bigcup_{j=1}^{n} \{y \in \langle a^2 \rangle x_j \mid B(x_i, y) \in \langle a^2 \rangle\} \right| \\
&= \frac{1}{|\langle a^2 \rangle|} |\{y \in V^\sharp \mid B(x_i, y) \in \langle a^2 \rangle\}| \\
&= \frac{1}{|\langle a^2 \rangle|} \sum_{b \in \langle a^2 \rangle} |\{y \in V \mid B(x_i, y) = b\}| \\
&= \frac{1}{|\langle a^2 \rangle|} |\langle a^2 \rangle| |\{y \in V \mid B(x_i, y) = 0\}| \\
&= q^{m-1}.
\end{aligned}$$

$$h \neq i \implies \sum_{j=1}^{n} W_{hj} W_{ij} = 0$$

can be proved in a similar manner.

$$\langle a^4 \rangle x_h = \langle a^4 \rangle x_i \implies W_{hj} W_{ij} = 0 \text{ for all } j.$$

If $\langle a^4 \rangle x_h \neq \langle a^4 \rangle x_i$, then

$$\begin{aligned} \#+ &= |\{j \mid W_{h,j} W_{i,j} = 1\}| \\ \#- &= |\{j \mid W_{h,j} W_{i,j} = -1\}| \end{aligned}$$

are both equal to

$$\frac{q^{m-2}(q-1)}{4}.$$

(ex. 1 if $q = 5$ and $m = 2$.)

Chan–Roger–Seberry (1986) missed:

$$
\begin{matrix}
+ & 0 & + & 0 & 0 & - & 0 & 0 & 0 & - & + & 0 \\
0 & + & + & + & + & + & 0 & 0 & 0 & 0 & 0 & 0 \\
+ & + & 0 & 0 & - & 0 & 0 & 0 & - & + & 0 & 0 \\
0 & + & 0 & 0 & 0 & - & - & 0 & + & 0 & - & 0 \\
0 & + & - & 0 & 0 & 0 & + & 0 & 0 & - & 0 & + \\
- & + & 0 & - & 0 & 0 & 0 & 0 & 0 & 0 & + & - \\
0 & 0 & 0 & - & + & 0 & - & 0 & - & 0 & 0 & + \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & - & - & - & - & - \\
0 & 0 & - & + & 0 & 0 & - & - & 0 & 0 & + & 0 \\
- & 0 & + & 0 & - & 0 & 0 & - & 0 & 0 & 0 & + \\
+ & 0 & 0 & - & 0 & + & 0 & - & + & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & + & - & + & - & 0 & + & 0 & 0
\end{matrix}
$$

Chan–Roger–Seberry (1986) missed:

+	0	+	0	0	−	0	0	0	−	+	0
0	+	+	+	+	+	0	0	0	0	0	0
+	+	0	0	−	0	0	0	−	+	0	0
0	+	0	0	0	−	−	0	+	0	−	0
0	+	−	0	0	0	+	0	0	−	0	+
−	+	0	−	0	0	0	0	0	0	+	−
0	0	0	−	+	0	−	0	−	0	0	+
0	0	0	0	0	0	0	−	−	−	−	−
0	0	−	+	0	0	−	−	0	0	+	0
−	0	+	0	−	0	0	−	0	0	0	+
+	0	0	−	0	+	0	−	+	0	0	0
0	0	0	0	+	−	+	−	0	+	0	0