# An Infinite Family of Weighing Matrices

Akihiro Munemasa[1]

[1]Graduate School of Information Sciences
Tohoku University

December 12, 2010
Kobe, Japan

# Definition

A weighing matrix $W$ of order $n$ and weight $k$ is an $n \times n$ matrix $W$ with entries $1, -1, 0$ such that

$$WW^T = kI_n,$$

where $I_n$ is the identity matrix of order $n$ and $W^T$ denotes the transpose of $W$.
We say that two weighing matrices $W_1$ and $W_2$ of order $n$ and weight $k$ are *equivalent* if there exist monomial matrices $P$ and $Q$ with $W_1 = PW_2Q$.

$k = n \implies$ Hadamard matrix
$k = 1 \implies$ Monomial matrix

# Chan–Roger–Seberry (1986)

Classified all weighing matrices of weight $k \leq 5$.

In particular, there are two weighing matrices of order $12$ and weight $5$, up to equivalence.

However, there is another one, discovered by Harada and A.M. recently, using the classification of self-dual codes of length $12$ over $\mathbb{F}_5$.

# Chan–Roger–Seberry (1986) missed:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| + | 0 | + | 0 | 0 | − | 0 | 0 | 0 | − | + | 0 |
| 0 | + | + | + | + | + | 0 | 0 | 0 | 0 | 0 | 0 |
| + | + | 0 | 0 | − | 0 | 0 | 0 | − | + | 0 | 0 |
| 0 | + | 0 | 0 | 0 | − | − | 0 | + | 0 | − | 0 |
| 0 | + | − | 0 | 0 | 0 | + | 0 | 0 | − | 0 | + |
| − | + | 0 | − | 0 | 0 | 0 | 0 | 0 | 0 | + | − |
| 0 | 0 | 0 | − | + | 0 | − | 0 | − | 0 | 0 | + |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | − | − | − | − | − |
| 0 | 0 | − | + | 0 | 0 | − | − | 0 | 0 | + | 0 |
| − | 0 | + | 0 | − | 0 | 0 | − | 0 | 0 | 0 | + |
| + | 0 | 0 | − | 0 | + | 0 | − | + | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | + | − | + | − | 0 | + | 0 | 0 |

# Inner product on $\mathbb{F}_5^2$ over $\mathbb{F}_5$

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2.$$

| | (0,1) | (1,0) | (1,1) | (1,2) | (1,3) | (1,4) |
|---|---|---|---|---|---|---|
| (0,1) | 1 | 0 | 1 | 2 | 3 | 4 |
| (1,0) | 0 | 1 | 1 | 1 | 1 | 1 |
| (1,1) | 1 | 1 | 2 | 3 | 4 | 0 |
| (1,2) | 2 | 1 | 3 | 0 | 2 | 4 |
| (1,3) | 3 | 1 | 4 | 2 | 0 | 3 |
| (1,4) | 4 | 1 | 0 | 4 | 3 | 2 |

We change

$$1 \to +, \quad 4 \to -, \quad 2 \to 0, \quad 3 \to 0$$

# Inner product on $\mathbb{F}_5^2$ over $\mathbb{F}_5$ $\to$ a $6 \times 6$ matrix with entries $0, \pm 1$

$A_1 =$

| | (0,1) | (1,0) | (1,1) | (1,2) | (1,3) | (1,4) |
|---|---|---|---|---|---|---|
| (0,1) | $+$ | 0 | $+$ | 0 | 0 | $-$ |
| (1,0) | 0 | $+$ | $+$ | $+$ | $+$ | $+$ |
| (1,1) | $+$ | $+$ | 0 | 0 | $-$ | 0 |
| (1,2) | 0 | $+$ | 0 | 0 | 0 | $-$ |
| (1,3) | 0 | $+$ | $-$ | 0 | 0 | 0 |
| (1,4) | $-$ | $+$ | 0 | $-$ | 0 | 0 |

# Inner product on $\mathbb{F}_5^2$ over $\mathbb{F}_5$ $\longrightarrow$ a $6 \times 6$ matrix with entries $0, \pm 1$

| | 2(0,1) | 2(1,0) | 2(1,1) | 2(1,2) | 2(1,3) | 2(1,4) |
|---|---|---|---|---|---|---|
| (0,1) | 2 | 0 | 2 | 4 | 1 | 3 |
| (1,0) | 0 | 2 | 2 | 2 | 2 | 2 |
| (1,1) | 2 | 2 | 4 | 1 | 3 | 0 |
| (1,2) | 4 | 2 | 1 | 0 | 4 | 3 |
| (1,3) | 1 | 2 | 3 | 4 | 0 | 1 |
| (1,4) | 3 | 2 | 0 | 3 | 1 | 4 |

Replace

$$1 \longrightarrow +, \quad 4 \longrightarrow -, \quad 2 \longrightarrow 0, \quad 3 \longrightarrow 0$$

to obtain $A_2$.

# Inner product on $\mathbb{F}_5^2$ over $\mathbb{F}_5$ $\rightarrow$ a $6 \times 6$ matrix with entries $0, \pm 1$

| | (0,1) | (1,0) | (1,1) | (1,2) | (1,3) | (1,4) |
|---|---|---|---|---|---|---|
| 2(0,1) | 2 | 0 | 2 | 4 | 1 | 3 |
| 2(1,0) | 0 | 2 | 2 | 2 | 2 | 2 |
| 2(1,1) | 2 | 2 | 4 | 1 | 3 | 0 |
| 2(1,2) | 4 | 2 | 1 | 0 | 4 | 3 |
| 2(1,3) | 1 | 2 | 3 | 4 | 0 | 1 |
| 2(1,4) | 3 | 2 | 0 | 3 | 1 | 4 |

Replace

$$1 \rightarrow +, \quad 4 \rightarrow -, \quad 2 \rightarrow 0, \quad 3 \rightarrow 0$$

to obtain $A_3$ (which is the same as $A_2$).

# Inner product on $\mathbb{F}_5^2$ over $\mathbb{F}_5$ $\longrightarrow$ a $6 \times 6$ matrix with entries $0, \pm 1$

| | 2(0,1) | 2(1,0) | 2(1,1) | 2(1,2) | 2(1,3) | 2(1,4) |
|---|---|---|---|---|---|---|
| 2(0,1) | 4 | 0 | 4 | 3 | 2 | 1 |
| 2(1,0) | 0 | 4 | 4 | 4 | 4 | 4 |
| 2(1,1) | 4 | 4 | 3 | 2 | 1 | 0 |
| 2(1,2) | 3 | 4 | 2 | 0 | 3 | 1 |
| 2(1,3) | 2 | 4 | 1 | 3 | 0 | 2 |
| 2(1,4) | 1 | 4 | 0 | 1 | 2 | 3 |

Replace

$$1 \longrightarrow +, \quad 4 \longrightarrow -, \quad 2 \longrightarrow 0, \quad 3 \longrightarrow 0$$

to obtain $A_4$ (which is the same as $-A_1$).

# Notation

- $q$: a prime power, $q \equiv 1 \pmod 4$ (ex. $q = 5$)
- $F = GF(q)$: a finite field, $F^\times = \langle a \rangle$ (ex. $a = 2$)
- $V$: a vector space of dimension $m+1$ over $F$, $m \geq 1$ (ex. $m = 1$)
- $V^\sharp = V \setminus \{0\}$
- $n = 2 \cdot (q^{m+1} - 1)/(q-1)$ (ex. $n = 12$)
- $X = V^\sharp / \langle a^2 \rangle = \{\langle a^2 \rangle x_i \mid 1 \leq i \leq n\}$ ($|X| = n$)
- $B : V \times V \to F$: nondegenerate bilinear form (ex. $B(x,y) = x_1 y_1 + x_2 y_2$)

Define $n \times n$ matrix $W$ by

$$W_{ij} = \begin{cases} 1 & \text{if } B(x_i, x_j) \in \langle a^4 \rangle, \text{ (ex. } \in \{1\}) \\ -1 & \text{if } B(x_i, x_j) \in a^2 \langle a^4 \rangle, \text{ (ex. } \in \{4\}) \\ 0 & \text{otherwise.} \end{cases}$$

# Main result

- $q$: a prime power, $q \equiv 1 \pmod 4$
- $F = GF(q)$: a finite field, $F^\times = \langle a \rangle$
- $V$: a vector space of dimension $m+1$ over $F$, $m \geq 1$
- $V^\sharp = V \setminus \{0\}$
- $n = 2 \cdot (q^{m+1} - 1)/(q-1)$
- $X = V^\sharp / \langle a^2 \rangle = \{\langle a^2 \rangle x_i \mid 1 \leq i \leq n\}$ $(|X| = n)$
- $B : V \times V \to F$: nondegenerate bilinear form

$$W_{ij} = \begin{cases} 1 & \text{if } B(x_i, x_j) \in \langle a^4 \rangle, \\ -1 & \text{if } B(x_i, x_j) \in a^2 \langle a^4 \rangle, \\ 0 & \text{otherwise.} \end{cases}$$

## Theorem

$W$ is a weighing matrix of order $n$ and weight $q^m$.

# Balanced generalized weighing (BGW) matrix

- $n, k$: positive integers
- $G$: multiplicatively written group
- $M$: $n \times n$ matrix with entries in $G \cup \{0\}$

$M$ is a balanced generalized weighing (BGW) matrix of order $n$, weight $k$ over $G$ if

- $\forall$ row has $k$ entries in $G$, $n - k$ entries $0$
- $|\{j \mid M_{hj}M_{ij}^{-1} = g,\ M_{hj} \neq 0,\ M_{ij} \neq 0\}|$ is a constant independent of $h, i$ (distinct) and $g \in G$.

$G = \{\pm 1\} \implies$ weighing matrix
$k = n \implies$ generalized Hadamard matrix

# Jungnickel–Tonchev (1999)

- $q$: prime power, $m \in \mathbb{N}$, $G = \mathrm{GF}(q)^\times$
- $\mathrm{Tr} : \mathrm{GF}(q^{m+1}) \to \mathrm{GF}(q)$
- $\mathrm{GF}(q^{m+1})^\times = \langle \alpha \rangle$, $n = \frac{q^{m+1}-1}{q-1}$
- $M = (\mathrm{Tr}(\alpha^{i+j}))_{0 \le i < n}$

Then $M$ is a BGW matrix of weight $q^m$

$M$: BGW matrix over $G$, $\chi : G \to H$ is a group epimorphism, then extending $\chi$ to $\chi : G \cup \{0\} \to H \cup \{0\}$
$\implies \chi(M)$ is a BGW matrix over $H$.
For a BGW matrix over $\mathrm{GF}(q)^\times$, one may take $\chi$ to be a multiplicative character.

# $q \equiv 1 \pmod 4$

- $q$: prime power, $m \in \mathbb{N}$, $G = \mathrm{GF}(q)^\times$
- $\mathrm{Tr} : \mathrm{GF}(q^{m+1}) \to \mathrm{GF}(q)$
- $\mathrm{GF}(q^{m+1})^\times = \langle \alpha \rangle$, $n = \frac{q^{m+1}-1}{q-1}$
- $M = (\mathrm{Tr}(\alpha^{i+j}))_{0 \le i < n}$
- $\chi : \mathrm{GF}(q)^\times \to \langle \sqrt{-1} \rangle = \{\pm 1, \pm\sqrt{-1}\}$: character of order $4$

Then $Z = \chi(M)$ is a BGW of order $n$, weight $q^m$ over $\langle \sqrt{-1} \rangle$

Write $Z = X + \sqrt{-1}Y$, where $X, Y$ are $(0,1)$-matrices.

$$W = \begin{bmatrix} X & Y \\ -Y & X \end{bmatrix}$$

is a weighing matrix of order $2n$, weight $q^m$.

# An easy proof

Let $Z$ be a BGW matrix of order $n$, weight $k$ over $\langle\sqrt{-1}\rangle$.
Then $Z = X + \sqrt{-1}Y \in M_n(\mathbb{C})$, where $X, Y$ are $(0,1)$-matrices.
Since $Z$ is a BGW matrix,

$$ZZ^* = kI$$

($1, -1, \sqrt{-1}, -\sqrt{-1}$ appear exactly the same number of times $\Longrightarrow$ inner product of rows $= 0$)

$$\Longrightarrow\ XX^T + YY^T = kI, \quad -XY^T + YX^T = 0$$

$$\Longrightarrow\ W = \begin{bmatrix} X & Y \\ -Y & X \end{bmatrix} \text{ satisfies } WW^T = kI.$$