Accumulation Points of the Smallest Eigenvalues of Graphs

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Eigenvalues of Graphs

- All graphs in this talks are finite, undirected and simple.
- *Eigenvalues* of a graph G are the eigenvalues of its *adjacency matrix* $A(G)$:

$$A(G)_{x,y} = \begin{cases} 1 & \text{if } x \text{ and } y \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

- $\mathrm{Spec}(G)$ = the multiset of eigenvalues of G.
- $\lambda_{\min}(G)$ = the smallest eigenvalue of G.
- $\lambda_{\max}(G)$ = the largest eigenvalue of G.
- The *degree* of a vertex x in G is the number of vertices adjacent to x, and is denoted by $d(x)$.
- $d_{\min}(G) \leq \overline{d}(G) \leq \lambda_{\max}(G) \leq d_{\max}(G)$.

Example: a path of length 2 

$\lambda_{\max}(P_2) = \sqrt{2}$:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} = \sqrt{2} \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}.$$

$\lambda_{\min}(P_2) = -\sqrt{2}$:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix} = -\sqrt{2} \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}.$$

$\lambda_{\min}(G)$ = the smallest eigenvalue

For bipartite graphs

- $\operatorname{Spec}(G) = -\operatorname{Spec}(G)$
- $\lambda_{\min}(G) = -\lambda_{\max}(G)$

In general, if we assume G is connected.

$$
\begin{aligned}
&G \text{ has at least } 2 \text{ vertices} \\
&\implies G \text{ has at least } 1 \text{ edge} \\
&\implies A(G) \supset \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ as a principal submatrix} \\
&\implies \lambda_{\min}(G) \leq -1.
\end{aligned}
$$

Equality holds if and only if G is complete.

$G_n = K_{n+2} -$ edge

- $K_n =$ complete graph with n vertices
- $J_n = n \times n$ matrix with all the entries 1
- $A(K_n) = J_n - I_n$.

$$A(G_n) = \begin{bmatrix} O_2 & \mathbf{1} \\ \mathbf{1} & A(K_n) \end{bmatrix}$$

$$\begin{aligned} \lambda_{\min}(G_n) &\geq \lambda_{\min}(A(G_n) - J_n) \\ &= \lambda_{\min}\left(\begin{bmatrix} -J_2 & O \\ O & -I_n \end{bmatrix}\right) \\ &= -2. \end{aligned}$$

In fact, $\lambda_{\min}(G_n) \to -2$ as $n \to \infty$, as we shall see. Set $\mu_n := -\lambda_{\min}(G_n) \leq 2$. Then $\mu_n > 1$.

$$-\lambda_{\min}(G) = \min\{\mu \in \mathbb{R}_{>0} \mid A(G) + \mu I_n \geq 0\}$$

$G_n = K_{n+2} -$ edge. $\mu_n := -\lambda_{\min}(G_n)$. Then $A(G_n) + \mu_n I_n \geq 0$, $1 < \mu_n \leq 2$.

$$\left[\begin{array}{cc|ccc} \mu_n & 0 & & & \\ 0 & \mu_n & & \mathbf{1} & \\ \hline & & \mu_n & & 1 \\ & \mathbf{1} & & \ddots & \\ & & 1 & & \mu_n \end{array}\right] \geq 0. \implies \left[\begin{array}{cc|c} \mu_n & 0 & n \\ 0 & \mu_n & n \\ \hline n & n & \tilde{n}^2 \end{array}\right] \geq 0$$

where $\tilde{n}$ is slightly larger than n; $\frac{\tilde{n}}{n} \to 1$.

$$\begin{bmatrix} 1 & 1 & -\frac{2}{n} \end{bmatrix} \left[\begin{array}{cc|c} \mu_n & 0 & n \\ 0 & \mu_n & n \\ \hline n & n & \tilde{n}^2 \end{array}\right] \begin{bmatrix} 1 \\ 1 \\ -\frac{2}{n} \end{bmatrix} \geq 0$$

implies $2\mu_n - 8 + 4\frac{\tilde{n}}{n} \geq 0$,

$$\lim_{n\to\infty} \lambda_{\min}(G_n) = \lim_{n\to\infty} (-\mu_n) \leq -2.$$

$G_n = K_{n+2} -$ edge

G_n and K_{n+2} differ only by an edge, but their smallest eigenvalues differ as $n \to \infty$:

$$\lambda_{\min}(G_n) \downarrow -2 \text{ while } \lambda_{\min}(K_{n+2}) = -1.$$

Note $d_{\min}(G_n) = n$.

Theorem (Hoffman (1977))

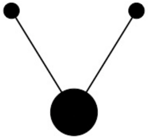
If $\{H_n\}_{n=1}^{\infty}$ is a sequence of graphs with $d_{\min}(H_n) \to \infty$, $\lambda = \lim_{n\to\infty} \lambda_{\min}(H_n)$ exists and $\lambda < -1$, then $\lambda \leq -2$.

Woo and Neumaier (1995)

Definition

A *Hoffman graph* $\mathfrak{H}$ is a graph (V, E) whose vertex set V consists of "slim" vertices and "fat" vertices, satisfying the following conditions:

1. every fat vertex is adjacent to at least one slim vertex,
2. fat vertices are pairwise non-adjacent.



$$A(\mathfrak{H}) = \begin{array}{c} \begin{array}{cc} \text{slim} & \text{fat} \end{array} \\ \begin{pmatrix} A & C \\ C^T & 0 \end{pmatrix} \end{array} = \left(\begin{array}{cc|c} 0 & 0 & 1 \\ 0 & 0 & 1 \\ \hline 1 & 1 & 0 \end{array}\right).$$

$$\lambda_{\min}(\mathfrak{H}) := -\min\{\mu \in \mathbb{R}_{>0} \mid \begin{pmatrix} A + \mu I & C \\ C^T & 0 \end{pmatrix} \geq 0\} = -2$$

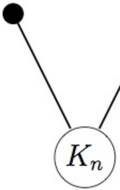
Hoffman's limit theorem

Theorem

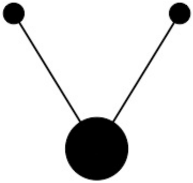
Let $\mathfrak{H}$ be a Hoffman graph. Let H_n be the ordinary graph obtained from $\mathfrak{H}$ by replacing every fat vertex f of $\mathfrak{H}$ by a n-clique $K(f)$, and joining all the neighbors of f with all the vertices of $K(f)$ by edges. Then

$$\lambda_{\min}(H_n) \geq \lambda_{\min}(\mathfrak{H}),$$
$$\lim_{n\to\infty} \lambda_{\min}(H_n) = \lambda_{\min}(\mathfrak{H}).$$

$H_n =$

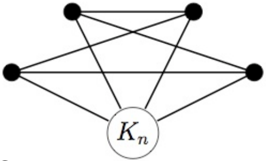


$\mathfrak{H} =$



$H_n = K_{n+4} - 2$ disjoint edges

The corresponding Hoffman graph has adjacency matrix



$$\left[\begin{array}{cc|c} O_2 & J_2 & \mathbf{1} \\ J_2 & O_2 & \mathbf{1} \\ \hline \mathbf{1} & \mathbf{1} & 0 \end{array}\right]$$

Since

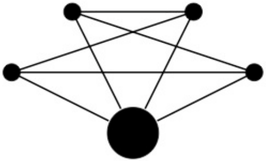
$$\begin{bmatrix} A + \mu I & C \\ C^T & 0 \end{bmatrix} \geq 0 \iff A + \mu I + CC^T \geq 0,$$

$$\lim_{n\to\infty} \lambda_{\min}(H_n) = \lambda_{\min}(\mathfrak{H}) = -\mu$$

where $\mu =$ smallest μ with

$$\left[\begin{array}{cc|c} \mu I_2 & O_2 & \mathbf{1} \\ O_2 & \mu I_2 & \mathbf{1} \\ \hline \mathbf{1} & \mathbf{1} & {\color{red}0} \end{array}\right] \geq 0 \iff \left[\begin{array}{c|c} \mu I_2 - J_2 & O_2 \\ \hline O_2 & \mu I_2 - J_2 \end{array}\right] \geq 0$$

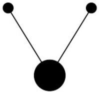
The smallest eigenvalue of the Hoffman graph



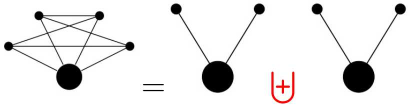
is $-\mu$, where μ is the smallest real number μ with

$$\left[\begin{array}{cc|c} \mu I_2 & O_2 & \mathbf{1} \\ O_2 & \mu I_2 & \mathbf{1} \\ \hline \mathbf{1} & \mathbf{1} & \color{red}{0} \end{array}\right] \geq 0 \iff \left[\begin{array}{c|c} \mu I_2 - J_2 & O_2 \\ \hline O_2 & \mu I_2 - J_2 \end{array}\right] \geq 0$$

$$\iff \mu I_2 - J_2 \geq 0 \iff \left[\begin{array}{c|c} \mu I_2 & \mathbf{1} \\ \hline \mathbf{1} & \color{red}{0} \end{array}\right] \geq 0$$

Same smallest eigenvalue as 

Sum



Definition

Let $\mathfrak{H}^1$ and $\mathfrak{H}^2$ be two non-empty induced Hoffman subgraphs of $\mathfrak{H}$. We write $\mathfrak{H} = \mathfrak{H}^1 \uplus \mathfrak{H}^2$, if

1. $V(\mathfrak{H}) = V(\mathfrak{H}^1) \cup V(\mathfrak{H}^2)$;
2. $\{V_s(\mathfrak{H}^1), V_s(\mathfrak{H}^2)\}$ is a partition of $V_s(\mathfrak{H})$;
3. if $x \in V_s(\mathfrak{H}^i)$, $y \in V_f(\mathfrak{H})$ and $x \sim y$, then $y \in V_f(\mathfrak{H}^i)$;
4. if $x \in V_s(\mathfrak{H}^1)$, $y \in V_s(\mathfrak{H}^2)$, then x and y have at most one common fat neighbor, and they have one if and only if they are adjacent.

If $\mathfrak{H} = \mathfrak{H}^1 \uplus \mathfrak{H}^2$ for some non empty subgraphs $\mathfrak{H}^1$ and $\mathfrak{H}^2$, then we call $\mathfrak{H}$ *decomposable*.

$\uplus$ and $\lambda_{\min}$

Theorem

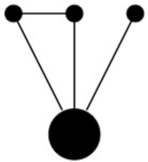
$$\lambda_{\min}(\mathfrak{H}^1 \uplus \mathfrak{H}^2) = \min\{\lambda_{\min}(\mathfrak{H}^1), \lambda_{\min}(\mathfrak{H}^2)\}.$$

- Because of this, the smallest eigenvalues of Hoffman graphs are easier to investigate than ordinary graphs.
- Hoffman graphs can be thought as sequence of ordinary graphs with increasing size of cliques.
- By Hoffman's limit theorem, the smallest eigenvalue of a Hoffman graph is a limit point of the smallest eigenvalues of ordinary graphs.
- There is no Hoffman graph with smallest eigenvalue between -1 and -2. The next largest possible smallest eigenvalue of a Hoffman graph is $-1-\sqrt{2}$.

$-1-\sqrt{2}$

Theorem (Hoffman (1977))

If $\{H_n\}_{n=1}^{\infty}$ is a sequence of graphs with $d_{\min}(H_n) \to \infty$, $\lambda = \lim\limits_{n\to\infty} \lambda_{\min}(H_n)$ exists and $\lambda < -2$, then $\lambda \leq -1-\sqrt{2}$.

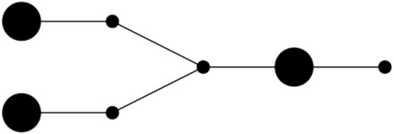


Theorem (Woo and Neumaier (1995))

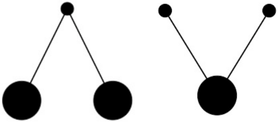
If $\{H_n\}_{n=1}^{\infty}$ is a sequence of graphs with $d_{\min}(H_n) \to \infty$, $\lambda = \lim\limits_{n\to\infty} \lambda_{\min}(H_n)$ exists and $\lambda < -1-\sqrt{2}$, then $\lambda \leq \alpha$, where α is the smallest root of x^3+2x^2-2x-2.

The smallest root of $x^3 + 2x^2 - 2x - 2$

is the eigenvalue of the Hoffman graph



Every Hoffman graph with smallest eigenvalue at least -2 is obtained by taking sums and subgraphs from just two Hoffman graphs, provided every slim vertex has at least one fat neighbor:



Every graph with smallest eigenvalue at least -2 with sufficiently large minimum degree is represented by a root system A_n or D_n.

Representation of a graph

Definition

A representation of norm m of a graph $G = (V, E)$ is a mapping $\phi : V \to \mathbb{R}^n$ such that

$$(\phi(x), \phi(y)) = \begin{cases} m & \text{if } x = y \in V, \\ 1 & \text{if } x \text{ and } y \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, G has a representation of norm m iff $\lambda_{\min}(G) \geq -m$.
So $\lambda_{\min}(G) \geq -2 \implies G$ is represented by a root system A_n, D_n or E_n.
We wish to investigate limit points of the smallest eigenvalues of graphs between -2 and -3. To do this, we need to investigate Hoffman graphs with the smallest eigenvalue between -2 and -3.

Theorem

Let $\mathfrak{H} = (V, E)$ be a Hoffman graph with $\lambda_{\min}(\mathfrak{H}) \geq -3$. Suppose

1. every slim vertex has at least one fat neighbor.
2. two distinct slim vertices have at most one common fat neighbor

Then there is a mapping $\phi : V_s \to \mathbb{R}^n$ such that $(\phi(x), \phi(y))$

$$= \begin{cases} 2 & \text{if } x = y\text{, and } x \text{ has a unique fat neighbor} \\ 1 & \text{if } x = y\text{, and } x \text{ has two fat neighbors} \\ 1 & \text{if } x \text{ and } y \text{ are adjacent, } \nexists \text{ common fat neighbor} \\ -1 & \text{if } x \text{ and } y \text{ are not adjacent, } \exists \text{ common fat neighbor} \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the image of ϕ generates a orthogonal direct sum of the standard lattices $\mathbb{Z}^n$ and root lattices A_n, D_n, E_n.