Def ① Hadamard matrix. $HH^T = nI$ $\quad H_{ij} = \pm 1$.

② Complex. Hadamard $HH^* = nI$ $\quad |H_{ij}| = 1$.

③ Type II (inverse orthogonal) matrix. $H(H^{(-)})^T = nI$. $\quad H_{ij} \neq 0$, $H_{ij} \in \mathbb{C}$.

H: type II $\Rightarrow$ λH is type II $\forall \lambda \neq 0$

Conj (Hadamard).

Example ① $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{\otimes n}$

② character table of abelian group $\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \bar\omega \\ 1 & \bar\omega & \omega \end{pmatrix}$. for $\mathbb{Z}_3$

③ $W = \begin{pmatrix} A & A & iH & -iH \\ A & A & -iH & iH \\ & & A & \end{pmatrix}$ $\quad A \in \langle I, J \rangle$

$A = I + u(J - I)$.

$u + u^{-1} = -n+2$.

$\exists u \in \mathbb{C}$. $\to$ [1.5]

Nomura (1994)

$$W = \begin{pmatrix} A & A & H & -H \\ A & A & -H & H \\ H^T & -H^T & A & A \\ -H^T & H^T & A & A \end{pmatrix}.$$

$4n \times 4n$

$A = u_0 I + u_1 (J - I) = u_0 (I + u(J - I))$

$= \begin{pmatrix} u_0 & & u_1 \\ & \ddots & \\ u_1 & & u_0 \end{pmatrix}$

H: order r.

$W = W^T$

$A = A^T$

WLOG $u_0 = 1$

Then $AA^{(-)} = nI \Leftrightarrow u_1 + u_1^{-1} = n$

$$\left(W^{(-)}\right)^T = \begin{pmatrix} A^{(-)} & A^{(-)} & H & -H \\ A^{(-)} & A^{(-)} & -H & H \\ H^T & -H^T & A^{(-)} & A^{(-)} \\ -H^T & H^T & & \end{pmatrix}$$

$\parallel$

$W^{(-)}$

$$AA^{(-)} = rI$$

$$AH - AH + HA^{(-)} - HA^{(-)} = 0.$$

"weaving" of Craigen (1995)

Nomura (1997)

W : type II $\in M_n(\mathbb{C})$

Define $Y_{ij} \in \mathbb{C}^n$ by $(Y_{ij})_k = \frac{W_{ki}}{W_{kj}}$.

$\left(Y_{ii} = \mathbb{1}. \quad J Y_{ij} = 0_{\ i \neq j} \right)$.

$N(W) = \{ X \in M_n(\mathbb{C}) \mid \forall ij \ \exists \theta_{ij} \in \mathbb{C} \quad X Y_{ij} = \theta_{ij} Y_{ij} \}$

Nomura algebra. (eigenvector)

$N(W)$ is closed under mltp. ← commutative

entrywise "

transpose

so $N(W) = \langle A_0 = I, A_1, \cdots, A_d \rangle$ (BM alg of A.S)

$A_i \circ A_j = \delta_{ij} A_i \quad \forall i \ \exists j \ A_i^T = A_j$.

$$A_1 = \begin{pmatrix} O & O & B & \bar{B} \\ O & O & \bar{B} & B \\ B^T & \bar{B}^T & 0 & 0 \\ \bar{B}^T & B^T & 0 & 0 \end{pmatrix}$$

$$\bar{B} = J - B$$

$$B = \frac{H+J}{2}.$$

(0,1)-matrix.

Then $A_1 \in N(W)$ (if u_0 is chosen appropriately).

$$\begin{pmatrix} \\ A_1 \\ \\ \end{pmatrix} \begin{matrix} \mathbb{1} + e_i e_j \\ \mathbb{1} + e_i - e_j \\ h \\ h. \end{matrix}$$

$$\begin{pmatrix} 0 & O & B & \bar{B} \\ O & O & \bar{B} & B \\ & & 0 & 0 \\ & & 0 & 0 \end{pmatrix} \begin{pmatrix} z_{ab}^{AA} \\ z_{ab}^{AA} \\ z_{ab}^{H^T H^T} \\ z_{ab}^{H^T H^T} \end{pmatrix}.$$

$B + \bar{B} = J$ (より).

O.

A_1 = adj matrix of a DRG

$\uparrow$ (0,1)-matrix

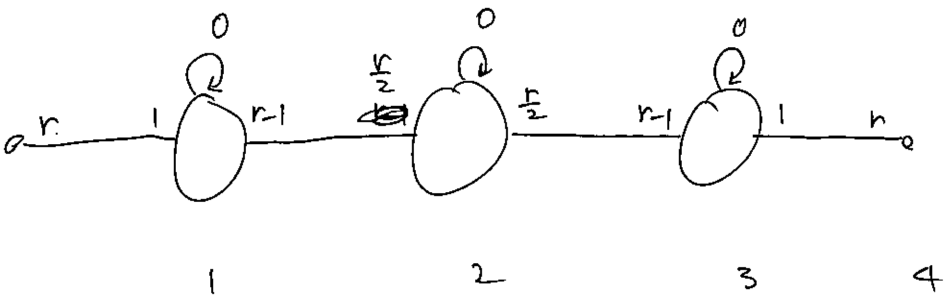
$$A_1^2 = rI + \frac{r}{2}A_2 \qquad A_2 = \begin{pmatrix} J-I & J-I & & \\ J-I & J-I & & O \\ & & J-I & J-I \\ & O & J-I & J-I \end{pmatrix}$$

$\cancel{BB^T + B^TB}$

$$A_1 A_2 = \cancel{\;}(r-1)A_1 + (r-1)A_3 \qquad A_3 = \begin{pmatrix} O & J \\ J & O \end{pmatrix} - A_1$$

$$A_1 A_3 \cancel{\Phi} = \frac{r}{2}A_2 + rA_4 \qquad = \begin{pmatrix} O & & \bar{B} & B \\ & & B & \bar{B} \\ {}^T & & O & \end{pmatrix} = A_1 A_4$$

$$A_4 = \begin{pmatrix} 0 & I & & \\ I & 0 & & O \\ & & 0 & I \\ & O & I & 0 \end{pmatrix}$$



JMN (1998)

$N(W) = \langle A_0, \cdots, A_4 \rangle$.

$\supset$ easy.

$\subset$ difficult.

JN (1999)

$$W' = \begin{pmatrix} A & A & H & -H \\ A & A & -H & H \\ -H^T & H^T & A & A \\ H^T & -H & A & A \end{pmatrix}.$$

$W' \neq W'^T$.
non symmetric

$$A_1' = \begin{pmatrix} O & & B & \bar{B} \\ & & \bar{B} & B \\ \bar{B}^T & B^T & O & \\ B^T & \bar{B}^T & & \end{pmatrix}$$

$A_1' \neq A_1'^T$.

not DRG.

$$A_3' = \begin{pmatrix} O & & \bar{B} & B \\ & & B & \bar{B} \\ & & O & \\ & & & \end{pmatrix}.$$

$A_1' + A_3' = \begin{pmatrix} O & J \\ J & O \end{pmatrix}$.

$A_0 = I$, A_2, A_4 as before.

$\langle A_0, A_1', A_2, A_3', A_4 \rangle$

BM algebra.

~~closed under entr~~

Theorem (Ikata-M.)

$N(W') = \langle \qquad \rangle$