Variations of two results of Jungnickel–Tonchev on projective spaces

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Two results of Jungnickel–Tonchev (1999, 2009)

- $\mathrm{PG}(d, q)$

(1999) Construction of a balanced generalized weighing matrix of order $\frac{q^{d+1}-1}{q-1}$, weight q^d

$\rightarrow$ weighing matrix of order $2\frac{q^{d+1}-1}{q-1}$, weight q^d if $q \equiv 1 \pmod 4$

- $\mathrm{PG}(2e, q)$

(2009) Distorting blocks of 2-$(\frac{q^{2e+1}-1}{q-1}, \frac{q^{e+1}-1}{q-1}, \begin{bmatrix} 2e-1 \\ e-1 \end{bmatrix})$ design

$\rightarrow$ twisted Grassmann graph of E. van Dam and J. Koolen (joint work with V. Tonchev).

Weighing matrices

Definition

A weighing matrix W of order n and weight k is an $n \times n$ matrix W with entries $1, -1, 0$ such that $WW^T = kI$.

- A Hadamard matrix is a $\mathrm{W}(n, n)$.
- We write "W is $\mathrm{W}(n, k)$" for short.
- $\mathrm{W}(n_1, k) \oplus \mathrm{W}(n_2, k) = \mathrm{W}(n_1 + n_2, k)$.

Chan–Rodger–Seberry (1985) classified weighing matrices of small n or k.
Harada–M. (to appear) extended classification, pointed out errors.
Notably, a $\mathrm{W}(12, 5)$ was missing, which is a signed incidence matrix of a semibiplane.

Balanced generalized weighing matrices

- G: a finite group (multiplicatively written), $\bar{G} = G \cup \{0\}$.

An $n \times n$ matrix $B = (b_{ij})$ with entries from $\bar{G}$ is a balanced generalized weighing matrix (written $\mathrm{BGW}(n, k, \mu)$) over G, if

- each row of B contains exactly k nonzero entries,
- for any $i \neq i'$, the multiset

$$\{g_{ij} g_{i'j}^{-1} \mid 1 \leq j \leq n,\ g_{ij} \neq 0,\ g_{i'j} \neq 0\}$$

represents every element of G exactly $\frac{\mu}{|G|}$ times.

If $G = \{\pm 1\}$, then $\mathrm{BGW}(n, k, \mu) \underset{\not\Longleftarrow}{\Longrightarrow} \mathrm{W}(n, k)$

If $G = \{1\}$, then W is just an incidence matrix of a symmetric 2-(n, k, μ) design.

$\mathrm{PG}(d,q)$

Jungnickel–Tonchev (1999) (also Jungnickel (1982)).

$$\exists\,\mathrm{BGW}\left(\frac{q^{d+1}-1}{q-1},\quad q^d,\quad q^d-q^{d-1}\right) \text{ over } \mathrm{GF}(q)^\times.$$

$\uparrow$ $|\,\mathrm{PG}(d,q)|$

$\uparrow$ Complements of hyperplanes

$\uparrow$ intersection

$G \longrightarrow 1$: incidence matrix of the symmetric design whose blocks are complements of hyperplanes.

Homomorphic image

- B: $\mathrm{BGW}(n,k,\mu)$ over G,
- $\chi : G \to H$: surjective homo. Define $\chi(0) = 0$.

Then $\chi(B)$: $\mathrm{BGW}(n,k,\mu)$ over H. In particular,

- $G \to \{\pm 1\}$ surjective $\implies$ $\mathrm{BGW}(n,k,\mu) \to \mathrm{W}(n,k)$.
- If $q \equiv 1 \pmod 4$, then $\mathrm{GF}(q)^\times \to \{\pm 1, \pm i\}$ (surjective).

$$\exists\,\mathrm{BGW}(\frac{q^{d+1}-1}{q-1}, q^d, q^d - q^{d-1}) \text{ over } \{\pm 1, \pm i\}$$

Doubling

Lemma

$B = X + iY$: $\mathrm{BGW}(n, k, \mu)$ over $\{\pm 1, \pm i\}$, where X and Y are $(0, \pm 1)$-matrices. Then

$$W = \begin{bmatrix} X & Y \\ -Y & X \end{bmatrix}$$

is a $\mathrm{W}(2n, k)$.

Proof.

In $M_n(\mathbb{C})$, $BB^* = kI \implies WW^T = kI$. □

Thus

$$\exists\, \mathrm{W}(2\frac{q^{d+1}-1}{q-1}, q^d) \quad \text{if } q \equiv 1 \pmod 4.$$

This gives the $\mathrm{W}(12, 5)$ missed by Chan–Rodger–Seberry.

Distorting blocks of $\mathrm{PG}_e(2e, q)$

- $V = V(2e+1, q)$, $\mathrm{PG}(2e, q) = \begin{bmatrix} V \\ 1 \end{bmatrix}$,
- Geometric design $\mathrm{PG}_e(2e, q)$ has blocks $\begin{bmatrix} V \\ e+1 \end{bmatrix}$.

$$2\text{-}\left(\frac{q^{2e+1}-1}{q-1}, \frac{q^{e+1}-1}{q-1}, \begin{bmatrix} 2e-1 \\ e-1 \end{bmatrix}\right) \text{ design.}$$

Distorting (Jungnickel–Tonchev, 2009): fix $H \in \begin{bmatrix} V \\ 2e \end{bmatrix}$ and a polarity σ on H (σ permutes $\begin{bmatrix} H \\ e \end{bmatrix}$).
For $W \in \begin{bmatrix} V \\ e+1 \end{bmatrix}$ with $W \cap H \in \begin{bmatrix} H \\ e \end{bmatrix}$, replace $W \cap H$ by $\sigma(W \cap H)$.
$\Longrightarrow$ 2-design with the same parameters but not isomorphic as the geometric design

The Grassmann graph $J_q(2e+1, e+1)$

Let $V = V(n, q)$. The Grassmann graph $J_q(n, d)$ has vertex set $= \begin{bmatrix} V \\ d \end{bmatrix}$. The adjacency is defined as follows:

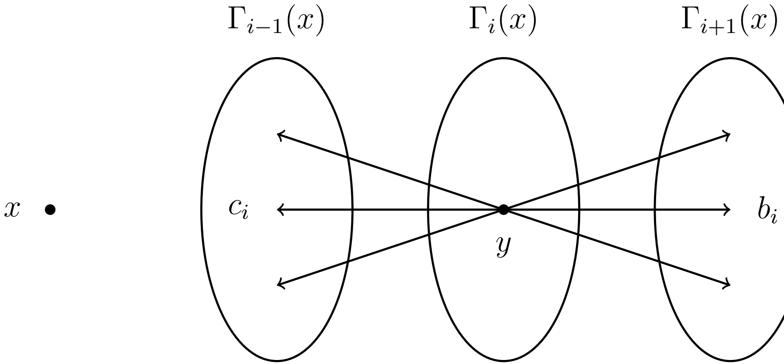
$$W_1 \sim W_2 \iff \dim W_1 \cap W_2 = d - 1.$$

Then $J_q(n, d)$ is a distance-transitive graph, with intersection array

$$b_i = q^{2i+1} \frac{(q^{d-i} - 1)(q^{n-d-i} - 1)}{(q-1)^2}, \qquad c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}^2.$$

The intersection array

$\Gamma_i(x) = \{\text{vertices at distance } i \text{ from } x\} \ni y$.
$c_i = |\Gamma_{i-1}(x) \cap \Gamma_1(y)|$. $b_i = |\Gamma_{i+1}(x) \cap \Gamma_1(y)|$.



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Characterization (Metsch, 1995): $J_q(n,d)$ is characterized by the intersection array, in many cases, but $(n,d) = (2e+1, e+1)$ was left open.

Twisted Grassmann graph (Van Dam–Koolen, 2005)

$V = V(2e+1,q)$, $H \in \begin{bmatrix} V \\ 2e \end{bmatrix}$. Define

$$\mathcal{A} = \{W \in \begin{bmatrix} V \\ e+1 \end{bmatrix} \mid W \not\subset H\},$$
$$\mathcal{B} = \begin{bmatrix} H \\ e-1 \end{bmatrix}.$$

The adjacency on $\mathcal{A} \cup \mathcal{B}$ is defined as follows:

$$W_1 \sim W_2 \iff \boxed{\dim W_1 \cap W_2 - \frac{1}{2}(\dim W_1 + \dim W_2) + 1} = 0.$$

This graph has the same intersection array as the Grassmann graph $J_q(2e+1,e+1)$ with vertex set $\begin{bmatrix} V \\ e+1 \end{bmatrix}$.

Blocks of the distorted design (Jungnickel–Tonchev, 2009)

Let σ be a polarity of H.
Points are $\mathrm{PG}(2e, q)$.
Blocks are

$$\mathcal{A}' = \{(W \setminus H) \cup \sigma(W \cap H) \mid W \in \begin{bmatrix} V \\ e+1 \end{bmatrix},\ W \not\subset H\},$$
$$\mathcal{B}' = \begin{bmatrix} H \\ e+1 \end{bmatrix}.$$

This design has the same parameters, q-rank, and block intersection numbers as the geometric design whose blocks are $\begin{bmatrix} V \\ e+1 \end{bmatrix}$.

The isomorphism

$$\mathcal{A} = \{W \in \begin{bmatrix} V \\ e+1 \end{bmatrix} \mid W \not\subset H\}, \qquad \mathcal{B} = \begin{bmatrix} H \\ e-1 \end{bmatrix},$$

$$\mathcal{A}' = \{(W \setminus H) \cup \sigma(W \cap H) \mid W \in \mathcal{A}\}, \quad \mathcal{B}' = \begin{bmatrix} H \\ e+1 \end{bmatrix}.$$

Lemma

Define $f : \mathcal{A} \cup \mathcal{B} \to \mathcal{A}' \cup \mathcal{B}'$ by

$$f(W) = \begin{cases} (W \setminus H) \cup \sigma(W \cap H) & \text{if } W \in \mathcal{A}, \\ \sigma(W) & \text{if } W \in \mathcal{B}. \end{cases}$$

Then for $W_1, W_2 \in \mathcal{A} \cup \mathcal{B}$, the blocks $f(W_1)$ and $f(W_2)$ meet at

$$\begin{bmatrix} \dim W_1 \cap W_2 - \frac{\dim W_1 + \dim W_2}{2} + 1 + e \\ 1 \end{bmatrix} \text{ points.}$$

The twisted Grassmann graph is the block graph

Theorem (M.–Tonchev)

The twisted Grassmann graph, is isomorphic to the block graph of the distorted design $(\mathrm{PG}(2e,q),\mathcal{A}'\cup\mathcal{B}')$, where two blocks are adjacent iff they have the largest possible intersection: $\begin{bmatrix} e \\ 1 \end{bmatrix}$.

Corollary

The automorphism group of the distorted design is the same as that of the twisted Grassmann graph, which is the stabilizer of H in $\mathrm{P\Gamma L}(2e+1,q)$.

Proof.

By Fujisaki–Koolen–Tagami (2006). □