Smallest eigenvalues of graphs and root systems of type A, D and E

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September 22, 2011
Topics in the Theory of Weyl Groups and Root Systems
in honor of Professor Jiro Sekiguchi on his 60th birthday
University of Tokyo

An enumeration problem related to the root system E_8

- E_8 has 240 roots.
- fix $\alpha \in E_8$, then $|\{\beta \in E_8 \mid (\alpha, \beta) = 1\}| = 56$.

$$\{\beta \in E_8 \mid (\alpha, \beta) = 1\}$$
$$= \{\beta_1, \alpha - \beta_1,\ \ \beta_2, \alpha - \beta_2,\ \ldots,\ \beta_{28}, \alpha - \beta_{28}\}.$$

Problem

Classify subsets Y of size 28 of the form $|Y \cap \{\beta_i, \alpha - \beta_i\}| = 1$ for all i, up to the action of $W(E_8)_\alpha$.

$$\left\lceil \frac{2^{28}}{|W(E_8)_\alpha|} \right\rceil = 93.$$

It turns out there are 467 orbits (about 10 seconds by MAGMA).

Fix $\alpha \in E_8$

$$\{\beta \in E_8 \mid (\alpha, \beta) = 1\}$$
$$= \{\beta_1, \alpha - \beta_1,\ \ \beta_2, \alpha - \beta_2,\ \ \ldots,\ \ \beta_{28}, \alpha - \beta_{28}\}.$$

Problem A

Classify subsets Y of size 28 of the form $|Y \cap \{\beta_i, \alpha - \beta_i\}| = 1$ for all i, up to the action of $W(E_8)$.

is a subproblem of

Problem B

Classify maximal subsets Y of E_8 satisfying $(\beta, \gamma) \geq 0$ for $\forall \beta, \gamma \in Y$.

The interest comes from graphs with smallest eigenvalue at least -2.
Kitazume–Munemasa (unpublished): via Problem A
Cvetković–Rowlinson–Simić (2004): not via Problem A

Eigenvalues of Graphs

A graph Γ (finite undirected simple) consists of a finite set of vertices V and edges E, where an edge is a 2-element subset of vertices.

The incidence matrix D of Γ:

rows: vertices

columns: edges

entries: 1 or 0 according to $v \in e$ or not.

The adjacency matrix of Γ:

$$A(\Gamma) = DD^T - \mathrm{Diag}((k_v)_{v \in V}) \quad A(\Gamma)_{u,v} = \begin{cases} 1 & \text{if } \{u,v\} \in E, \\ 0 & \text{otherwise,} \end{cases}$$

where $k_v = |\{e \in E \mid v \in e\}|$: the degree of v.

Leonhard Euler and William T. Tutte

Eulerian graphs (Seven bridges of Königsberg, 1735) (from Wikipedia)

Bill Tutte (1917–2002): modern graph theory
http://www.math.uwaterloo.ca/

Dragomir Djokovic (retired in 2006)

Alan J. Hoffman (1924–)

$A(\Gamma)$: adjacency matrix of Γ

$\lambda_{\min}(A(\Gamma)) \geq -2$

$\iff A(\Gamma) + 2I \geq 0$

Gram matrix of a set of vectors of norm 2

http://www.research.ibm.com/people/a/ajh/

founder of "Linear Algebra and Applications"

graphs with smallest eigenvalue at least -2: combinatorial characterization: generalized line graphs, finitely many exceptional graphs

Cameron–Goethals–Seidel–Shult (1976): Root system of type A, D (generalized line graphs) or E (exceptional graphs)

Representation of graphs

Let $\Gamma = (V, E)$ be a graph, $m \in \mathbb{R}$, $m > 1$.
A representation of norm m of Γ is a mapping $\phi : V \to \mathbb{R}^n$ such that

$$(\phi(u), \phi(v)) = \begin{cases} m & \text{if } u = v, \\ 1 & \text{if } \{u, v\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Gram matrix $A(\Gamma) + mI$.

$\exists$ representation of norm $m \iff \lambda_{\min}(A(\Gamma)) \geq -m$.

For $m = 2$: root system of type A,D or E.

From -2 to -3

- What can we say about a finite subset X of $\mathbb{R}^n$ satisfying

$$\forall \alpha, \beta \in X,\ (\alpha, \beta) = \begin{cases} \pm 3 & \text{if } \alpha = \pm\beta, \\ \pm 1 \text{ or } 0 & \text{otherwise.} \end{cases}$$

- Graphs with smallest eigenvalue at least -3?
- Integral lattices generated by a set of vectors of norm 3?

$-2 > -1-\sqrt{2} \approx -2.4142 > -3$

Hoffman (1977): The problem does not immediately get wild if we go beyond -2.

Theorem

$-2 > \forall\theta > -1-\sqrt{2},\ \exists d > 0,$
there is no graph Δ with minimum degree $> d$ and $-2 > \lambda_{\min}(\Delta) \geq \theta$.

Not true for $\theta = -1-\sqrt{2}$.