Super Catalan numbers and Krawtchouk polynomials

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Binomial coefficients

$$\binom{n}{r} = \frac{n(n-1)\cdots(n-r+1)}{r(r-1)\cdots 1} = \frac{n!}{r!(n-r)!}$$

is an integer, because it counts the number of r-subsets of an n-set. The middle binomial coefficient

$$\binom{2n}{n} = \frac{(2n)!}{n!n!}$$

is not only an integer, but also divisible by $n+1$. That is, the Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}$$

is an integer, because. . . .

Catalan numbers

The Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}$$

is an integer, because it counts the number of different ways $n+1$ factors can be completely parenthesized:

$$((ab)c)d, (a(bc))d, (ab)(cd), a((bc)d), a(b(cd)).$$

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}.$$

Super Catalan numbers

Catalan (1874) also observed

$$S(m,n) = \frac{(2m)!(2n)!}{m!n!(m+n)!} \in \mathbb{Z}.$$

$$S(0,n) = \binom{2n}{n}, \quad S(1,n) = 2C_n = \frac{2}{n+1}\binom{2n}{n}.$$

Gessel–Xin (2005) gave a combinatorial reason for $S(2,n), S(3,n) \in \mathbb{Z}$.
On the other hand, von Szily's identity (1894)

$$S(m,n) = \sum_{k\in\mathbb{Z}} (-1)^k \binom{2m}{m+k}\binom{2n}{n-k}$$

implies that $S(m,n) \in \mathbb{Z}$.

Combinatorial interpretation of von Szily's identity

$$
\begin{aligned}
S(m,n) &= \sum_{k\in\mathbb{Z}} (-1)^k \binom{2m}{m+k}\binom{2n}{n-k} \\
&= (-1)^m \sum_{h=0}^{m+n} (-1)^h \binom{2m}{h}\binom{2n}{m+n-h} \quad (h=m+k)
\end{aligned}
$$

$$
S(m,n) = (-1)^m \sum_{h=0}^{m+n} (-1)^h \left| \left\{ X \;\middle|\; \begin{array}{l} X \subset M \cup N \\ |X| = m+n \\ |X \cap M| = h \end{array} \right\} \right|,
$$

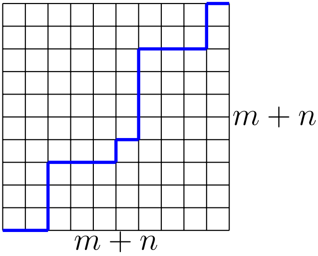
where $|M| = 2m$, $|N| = 2n$, $M \cap N = \emptyset$.

Lattice paths

Let $|M| = 2m$, $|N| = 2n$, $M \cap N = \emptyset$, so $|M \cup N| = 2(m+n)$.

$$\left| \left\{ X \;\middle|\; \begin{array}{l} X \subset M \cup N \\ |X| = m+n \end{array} \right\} \right|$$

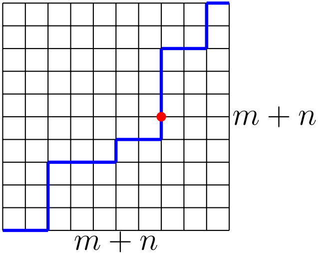
counts the number of all lattice paths from $(0,0)$ to $(m+n, m+n)$ consisting of unit steps $\rightarrow$ or $\uparrow$.



$|M| = 2m,\ |N| = 2n,\ M \cap N = \emptyset$

$$\left| \left\{ X \;\middle|\; \begin{array}{l} X \subset M \cup N \\ |X| = m+n \\ |X \cap M| = h \end{array} \right\} \right|$$

counts the number of all lattice paths from $(0,0)$ to $(m+n, m+n)$ consisting of unit steps $\rightarrow$ or $\uparrow$, such that the height is h after the $2m$-th step.



$$m = 6$$
$$n = 4$$
$$2m = 12$$
$$h = 5$$

$|M| = 2m,\ |N| = 2n,\ M \cap N = \emptyset$

$$
\begin{aligned}
&S(m,n)\\
&= (-1)^m \sum_{h=0}^{m+n} (-1)^h \binom{2m}{h} \binom{2n}{m+n-h}\\
&= (-1)^m \sum_{h=0}^{m+n} (-1)^h \left| \left\{ X \;\middle|\; \begin{array}{l} X \subset M \cup N \\ |X| = m+n \\ |X \cap M| = h \end{array} \right\} \right|\\
&= (-1)^m \sum_{h=0}^{m+n} (-1)^h \left| \left\{ \begin{array}{l} \text{lattice paths from } (0,0) \text{ to} \\ (m+n, m+n) \text{ consisting of unit} \\ \text{steps} \rightarrow \text{or} \uparrow\text{, such that the height} \\ \text{is } h \text{ after the } 2m\text{-th step} \end{array} \right\} \right|
\end{aligned}
$$

$|M| = 2m,\ |N| = 2n,\ M \cap N = \emptyset$

Let $\mathbb{F}_2 = \{0, 1\}$ denote the finite field with two elements (equipped with binary addition and multiplication). For a vector $\boldsymbol{x} = (x_1, \dots, x_d) \in \mathbb{F}_2^d$,

$$\begin{aligned} \operatorname{supp}(\boldsymbol{x}) &= \{i \mid 1 \le i \le d,\ x_i = 1\}, \\ \operatorname{wt}(\boldsymbol{x}) &= |\operatorname{supp}(\boldsymbol{x})| \end{aligned}$$

Let $\boldsymbol{z} = (1, \dots, 1, 0, \dots, 0)$, $\operatorname{supp}(\boldsymbol{z}) = 2m$. Then

$$\left| \left\{ X \,\middle|\, \begin{array}{l} X \subset M \cup N \\ |X| = m + n \\ |X \cap M| = h \end{array} \right\} \right| = \sum_{\substack{\boldsymbol{x} \in \mathbb{F}_2^{2(m+n)} \\ \operatorname{wt}(\boldsymbol{x}) = m+n \\ \operatorname{supp}(\boldsymbol{x}) \cap \operatorname{supp}(\boldsymbol{z}) = h}} 1$$

counts the number of all binary vectors $\boldsymbol{x} \in \mathbb{F}_2^{2(m+n)}$ of weight $m + n$, such that $\operatorname{supp}(\boldsymbol{x}) \cap \operatorname{supp}(\boldsymbol{z}) = h$.

$$S(m,n) = (-1)^m \sum_{h=0}^{m+n} (-1)^h \left| \left\{ X \;\middle|\; \begin{array}{l} X \subset M \cup N \\ |X| = m+n \\ |X \cap M| = h \end{array} \right\} \right|$$

$$= (-1)^m \sum_{h=0}^{m+n} (-1)^{h} \sum_{\substack{\boldsymbol{x} \in \mathbb{F}_2^{2(m+n)} \\ \mathrm{wt}(\boldsymbol{x}) = m+n \\ |\mathrm{supp}(\boldsymbol{x}) \cap \mathrm{supp}(\boldsymbol{z})| = h}} 1$$

$$= (-1)^m \sum_{h=0}^{m+n} \sum_{\substack{\boldsymbol{x} \in \mathbb{F}_2^{2(m+n)} \\ \mathrm{wt}(\boldsymbol{x}) = m+n \\ |\mathrm{supp}(\boldsymbol{x}) \cap \mathrm{supp}(\boldsymbol{z})| = h}} (-1)^{|\mathrm{supp}(\boldsymbol{x}) \cap \mathrm{supp}(\boldsymbol{z})|}$$

$$= (-1)^m \sum_{\substack{\boldsymbol{x} \in \mathbb{F}_2^{2(m+n)} \\ \mathrm{wt}(\boldsymbol{x}) = m+n}} (-1)^{\langle \boldsymbol{x}, \boldsymbol{z} \rangle} \qquad \left(\langle \boldsymbol{x}, \boldsymbol{z} \rangle = \sum x_i z_i\right)$$

Krawtchouk polynomials

$$(-1)^m S(m,n) = \sum_{\substack{\boldsymbol{x}\in\mathbb{F}_2^{2(m+n)} \\ \mathrm{wt}(\boldsymbol{x})=m+n}} (-1)^{\langle \boldsymbol{x},\boldsymbol{z}\rangle}.$$

Krawtchouk polynomial $K_j^d(x)$ is defined by

$$K_j^d(z) = \sum_{\substack{\boldsymbol{x}\in\mathbb{F}_2^{d} \\ \mathrm{wt}(\boldsymbol{x})=j}} (-1)^{\langle \boldsymbol{x},\boldsymbol{z}\rangle}, \quad \text{where } \mathrm{wt}(\boldsymbol{z}) = z.$$

$$= \sum_{h=0}^{j} (-1)^h \binom{z}{h}\binom{d-z}{j-h}.$$

Then

$$(-1)^m S(m,n) = K_{m+n}^{2(m+n)}(2m).$$

$(-1)^m S(m,n) = K_{m+n}^{2(m+n)}(2m)$

Krawtchouk polynomials are the eigenvalues of the distance-j graph of the d-cube. More precisely,

$$\{(-1)^m S(m,n) \mid m,n \geq 0,\ m+n=d\} \cup \{0\}$$

coincides with the set of eigenvalues of the distance-d graph of the $2d$-cube, which is known as the orthogonality graph.

$$\text{vertices} = \{\pm 1\}^{2d},$$
$$\boldsymbol{x} \sim \boldsymbol{y} \iff \langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0.$$

MacWilliams identities $C \subset \mathbb{F}_2^d$, $C^\perp \subset \mathbb{F}_2^d$

$$|\{\boldsymbol{x} \in C^\perp \mid \mathrm{wt}(\boldsymbol{x}) = j\}|$$

$$= \sum_{\substack{\boldsymbol{x}\in\mathbb{F}_2^d \\ \mathrm{wt}(\boldsymbol{x})=j}} \frac{1}{|C|} \sum_{\boldsymbol{z}\in C} (-1)^{\langle \boldsymbol{x},\boldsymbol{z}\rangle} \qquad \left(\begin{cases} 1 & \boldsymbol{x} \in C^\perp, \\ 0 & \boldsymbol{x} \notin C^\perp \end{cases} \right)$$

$$= \frac{1}{|C|} \sum_{z=0}^{d} \sum_{\substack{\boldsymbol{z}\in C \\ \mathrm{wt}(\boldsymbol{z})=z}} \sum_{\substack{\boldsymbol{x}\in\mathbb{F}_2^d \\ \mathrm{wt}(\boldsymbol{x})=j}} (-1)^{\langle \boldsymbol{x},\boldsymbol{z}\rangle}$$

$$= \frac{1}{|C|} \sum_{z=0}^{d} \sum_{\substack{\boldsymbol{z}\in C \\ \mathrm{wt}(\boldsymbol{z})=x}} K_j^d(z) \qquad \left(K_j^d(z) = \sum_{\substack{\boldsymbol{x}\in\mathbb{F}_2^d \\ \mathrm{wt}(\boldsymbol{x})=j}} (-1)^{\langle \boldsymbol{x},\boldsymbol{z}\rangle} \right)$$

$$= \frac{1}{|C|} \sum_{z=0}^{d} K_j^d(z) |\{\boldsymbol{z} \in C \mid \mathrm{wt}(\boldsymbol{z}) = z\}|.$$

$S(m,n)$ is the size of a set?

$$S(m,n) = (-1)^m \sum_{h=0}^{m+n} (-1)^h \left| \left\{ X \;\middle|\; \begin{array}{l} X \subset M \cup N \\ |X| = m+n \\ |X \cap M| = h \end{array} \right\} \right|$$

$$= \sum_{\substack{0 \le h \le m+n \\ h+m\text{: even}}} \left| \left\{ X \;\middle|\; \begin{array}{l} X \subset M \cup N \\ |X| = m+n \\ |X \cap M| = h \end{array} \right\} \right|$$

$$- \sum_{\substack{0 \le h \le m+n \\ h+m\text{: odd}}} \left| \left\{ X \;\middle|\; \begin{array}{l} X \subset M \cup N \\ |X| = m+n \\ |X \cap M| = h \end{array} \right\} \right|$$

$S(m,n)$ is the size of a set?

Problem

Find an injection from

$$\bigcup_{\substack{0\le h\le m+n\\ h+m:\ \text{odd}}} \left\{ X \,\middle|\, \begin{array}{l} X \subset M \cup N \\ |X| = m+n \\ |X \cap M| = h \end{array} \right\}$$

to

$$\bigcup_{\substack{0\le h\le m+n\\ h+m:\ \text{even}}} \left\{ X \,\middle|\, \begin{array}{l} X \subset M \cup N \\ |X| = m+n \\ |X \cap M| = h \end{array} \right\}$$

and describe the complement of the image.