Graphs with complete multipartite μ-graphs

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joint work with A. Jurišić and Y. Tagami
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Distance-Regular Graphs

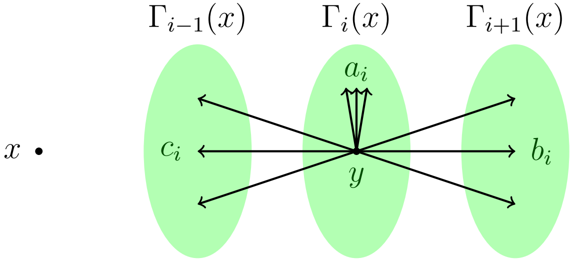
Brouwer–Cohen–Neumaier (1988).
Examples: Dual polar spaces $= \{$max. totally isotropic subsp.$\}$
and their subconstituent: eg. alternating forms graph.

Main Problem: Classify distance-regular graphs.

- classification of feasible parameters
- characterization by parameters
- characterization by local structure

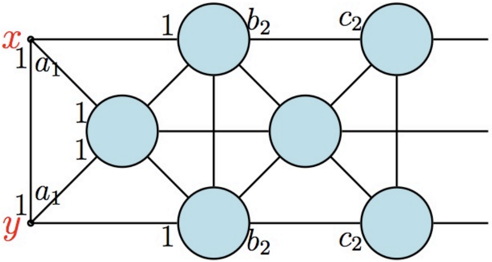
A local characterization of the graphs of alternating forms and the graphs of quadratic forms graphs over $\mathrm{GF}(2)$
A. Munemasa, D.V. Pasechnik, S.V. Shpectorov

Definition of a distance-regular graph



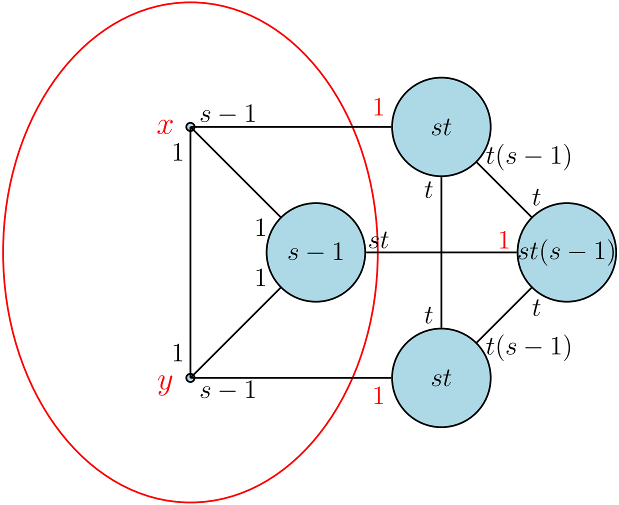
- $\Gamma_i(x)$: the set of vertices at distance i from x
- the numbers a_i, b_i, c_i are independent of x and $y \in \Gamma_i(x)$.
- a_i, b_i, c_i are called the parameters of a distance-regular graph Γ.

1-Homogeneity



- Nomura (1987) obtained inequalities among a_i, b_i, c_i
- requiring constant number of edges between cells is an additional condition (1-homogeneity).

Generalized Quadrangle of order (s,t)



Local Characterization of Alternating Forms Graph $\mathrm{Alt}(n, 2)$ over $\mathrm{GF}(2)$

Local Graph $= \Gamma(x) =$ neighborhood of x. Assume that a distance-regular graph Γ has the same local graph as $\mathrm{Alt}(n, 2)$, i.e., Grassmann graph ($=$ line graph of $\mathrm{PG}(n-1, 2)$), and the same parameters (in particular $c_2 = \mu = 20$). Then $\Gamma \cong \mathrm{Alt}(n, 2)$ or $\mathrm{Quad}(n-1, 2)$ (M.–Shpectorov–Pasechnik).

Key idea: "μ local = local μ", where "$\mu = \Gamma(x) \cap \Gamma(y)$" with $y \in \Gamma_2(x)$. Taking $z \in \Gamma(x) \cap \Gamma(y)$,

$$\begin{aligned}
\mu \text{ of local of } \Gamma &= \mu \text{ of } \Gamma(z) \\
&= (\Gamma(x) \cap \Gamma(y)) \cap \Gamma(z) \\
&= \Gamma(z) \cap (\Gamma(x) \cap \Gamma(y)) \\
&= \text{local of } \Gamma(x) \cap \Gamma(y) \\
&= \text{local of } \mu \text{ of } \Gamma.
\end{aligned}$$

Local Characterization of Alternating Forms Graph $\mathrm{Alt}(n,2)$ over $\mathrm{GF}(2)$

If local graphs of Γ are Grassmann (line graph of $\mathrm{PG}(n-1,2)$), then "μ local = local μ" implies

$$\mu \text{ of Grassmann} = \text{ local of } \mu$$

hence

$$3 \times 3 \text{ grid} = \text{ local of } \mu$$

μ-graphs of Γ are locally 3×3-grid, and $\mu = c_2 = 20 = \binom{6}{3}$ $\implies J(6,3)$.

Juršić–Koolen, 2003

From now on, a μ-graph of a graph is the subgraph induced on the set of common neighbors of two vertices at distance 2.

$$\begin{aligned}\text{Cocktail party graph} &= \text{complete graph } K_{2p} \text{ minus a matching} \\ &= \text{complete multipartite graph } K_{p\times 2} \\ &\quad (p \text{ parts of size } 2)\end{aligned}$$

Classified 1-homogeneous distance-regular graphs with cocktail party μ-graph $K_{p\times 2}$ with $p \geq 2$.

Examples

					μ-graph
	$K_{p\times 2}$				
	$\vdots$				
	$K_{6\times 2}$	Gosset			$K_{5\times 2}$
local $\downarrow$	$K_{5\times 2}$	Schläfli			$K_{4\times 2}$
	$K_{4\times 2}$	$\frac{1}{2}$5-cube	$\frac{1}{2}n$-cube		$K_{3\times 2}$
	$K_{3\times 2}$	$J(5,2)$	$J(n,2)$	$J(n,k)$	$K_{2\times 2}$
	$K_{2\times 2}$	2×3	$2\times(n-2)$	$k\times(n-k)$	$K_{1\times 2}$

The bottom rows are all grids.
Jurišić–Koolen (2007): 1-homogeneous distance-regular graphs with cocktail party μ-graph $K_{p\times 2}$ with $p\geq 2$ are contained in those shown above and some of their quotients.

Juršić–Koolen, 2007

Complete multipartite graph $K_{p\times n}$ is a generalization of cocktail party graph $K_{p\times 2}$.
Examples

$\vdots$					μ-graph
$K_{6\times n}$					$K_{5\times n}$
$K_{5\times n}$	$3.O_7(3)$				$K_{4\times n}$
$K_{4\times n}$	$O_6^+(3)$	Meixner			$K_{3\times n}$
$K_{3\times n}$	$O_5(3)$	$U_5(2)$	Patterson	$3.O_6^-(3)$	$K_{2\times n}$
$K_{2\times n}$	$\mathrm{GQ}(2,2)$	$\mathrm{GQ}(3,3)$	$\mathrm{GQ}(9,3)$	$\mathrm{GQ}(4,2)$	$K_{1\times n} = \overline{K_n}$
					$n = t+1$

They assumed distance-regularity, but having $K_{p\times n}$ as μ-graphs turns out to be a very strong restriction already.

"local $\mu = \mu$ local"

In local characterization,

$$\underset{\text{known}}{\underset{\uparrow}{\text{local of}}} \quad \underset{\text{derive}}{\underset{\uparrow}{\mu\text{-graph}}} \quad = \quad \underset{\text{known}}{\underset{\uparrow}{\mu \text{ of}}} \quad \underset{\text{assume}}{\underset{\uparrow}{\text{local}}}$$

In μ characterization,

$$\underset{\text{known}}{\underset{\uparrow}{\text{local of}}} \quad \underset{\text{assume}}{\underset{\uparrow}{\mu\text{-graph}}} \quad = \quad \underset{\text{known}}{\underset{\uparrow}{\mu \text{ of}}} \quad \underset{\text{derive}}{\underset{\uparrow}{\text{local}}}$$

Example

$$\underset{K_{(p-1)\times n}}{\underset{\uparrow}{\text{local of}}} \quad \underset{\text{assume}}{\underset{\uparrow}{\mu = K_{p\times n}}} \quad = \quad \underset{K_{(p-1)\times n}}{\underset{\uparrow}{\mu \text{ of}}} \quad \underset{\text{derive}}{\underset{\uparrow}{\text{local}}}$$

Taking local, $\mu = K_{p\times n} \to \mu = K_{(p-1)\times n}$

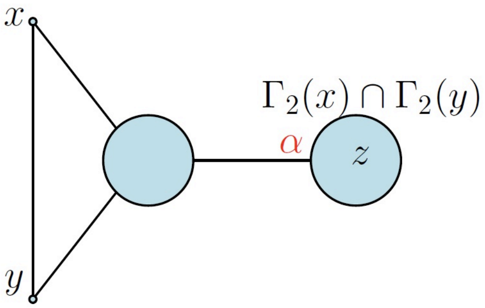
Assume every μ-graph of Γ is $K_{p\times n}$. Taking local graph $(p-1)$ times, one obtains a graph Δ whose μ-graphs are $K_{1\times n} = \overline{K_n}$: equivalently, $\nexists K_{1,1,2}$,

$$\forall \text{ edge} \subset \exists! \text{maximal clique}$$

Such graphs always come from a geometric graph such as GQ?

					μ-graph
$\vdots$					
$K_{6\times n}$					$K_{5\times n}$
$K_{5\times n}$	$3.O_7(3)$				$K_{4\times n}$
$K_{4\times n}$	$O_6^+(3)$	Meixner			$K_{3\times n}$
$K_{3\times n}$	$O_5(3)$	$U_5(2)$	Patterson	$3.O_6^-(3)$	$K_{2\times n}$
$K_{2\times n}$	$\mathrm{GQ}(2,2)$	$\mathrm{GQ}(3,3)$	$\mathrm{GQ}(9,3)$	$\mathrm{GQ}(4,2)$	$K_{1\times n} = \overline{K_n}$
					$n = t+1$

The parameter α

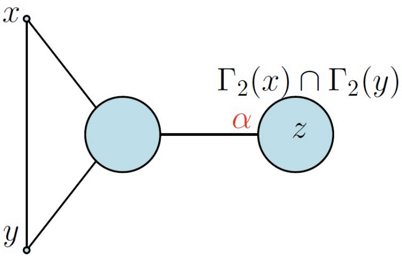


For a graph Γ, we say the parameter α exists if $\exists x, y, z$,

$$d(x,y) = 1,\ d(x,z) = d(y,z) = 2$$

and $|\Gamma(x) \cap \Gamma(y) \cap \Gamma(z)| = \alpha(\Gamma)$ for all such x, y, z.
Example: $\alpha(\mathrm{GQ}(s,t)) = 1$ if $s, t \geq 2$.

α-graph is a clique, hence $\alpha \le p$



Suppose every μ-graph of Γ is $K_{p\times n}$,and α exists.

Claim: $\Gamma(x) \cap \Gamma(y) \cap \Gamma(z)$ is a clique. Indeed, if nonadjacent $u, v \in \Gamma(x) \cap \Gamma(y) \cap \Gamma(z)$, then $x, y, z \in \Gamma(u) \cap \Gamma(v) \cong K_{p\times n}$, but

$$d(x, y) = 1,\ d(x, z) = d(y, z) = 2: \text{ contradiction.}$$

$\alpha(\Gamma)$ is bounded by the clique size in $\Gamma(x) \cap \Gamma(z) \cong K_{p\times n}$ which is p.

The parameter α

We have shown $\alpha(\Gamma) \leq p$.

- One can also shows $\alpha(\Gamma) \geq p - 1$.
- If Δ is a local graph, then $\alpha(\Delta)$ exists and $\alpha(\Delta) = \alpha(\Gamma) - 1$.

Regularity

Lemma

Let Γ be a connected graph, M a non-complete graph. Assume every μ-graph of Γ is M. Then Γ is regular.

Proof.

By two-way counting (BCN, p.4, Proposition 1.1.2.) □

Lemma

Let Γ be graph, M a graph without isolated vertex. Assume every μ-graph of Γ is M. Then every local graph of M has diameter 2.

Reduction

Lemma

Let Γ be a connected graph. Assume every μ-graph of Γ is $K_{p\times n}$, and α exists. Let Δ be a local graph of Γ. Then

- Γ is regular,
- Δ has diameter 2,
- every μ-graph of Δ is $K_{(p-1)\times n}$.
- $\alpha(\Delta)$ exists and $\alpha(\Delta) = \alpha(\Gamma) - 1$.

We know $\alpha(\Gamma) = p$ or $p - 1$.
Suggests that the reverse procedure of taking a local graph does not seem possible so many times, meaning p cannot be too large.

Main Result

Theorem

Let Γ be a connected graph. Assume every μ-graph of Γ is $K_{p\times n}$, where $p, n \geq 2$, and α exists in Γ. Then

(i) $p = \alpha(\Gamma)$ unless $(p, \alpha(\Gamma)) = (2, 1)$ and diameter ≥ 3.

(ii) If $n \geq 3$, then

$$p = \alpha(\Gamma) = 2 \implies \Gamma \text{ locally } \mathrm{GQ}(s, n-1),$$
$$p = \alpha(\Gamma) = 3 \implies \Gamma \text{ locally}^2 \ \mathrm{GQ}(n-1, n-1),$$
$$p = \alpha(\Gamma) = 4 \implies \Gamma \text{ locally}^3 \ \mathrm{GQ}(2, 2),$$

$p \geq 5$: impossible.

proof of (i).

Rule out $(p, \alpha(\Gamma)) = (2, 1)$ when diameter$= 2$ (strongly regular). □

Open Problem

Rule out $(p, \alpha(\Gamma)) = (2, 1)$ when diameter≥ 3.

This might occur even when $n = 2$: μ-graph of Γ is cocktail party graph $K_{2\times 2} = C_4$. Nonexistence was conjectured by Jurišić–Koolen (2003).