Graphs with Smallest Eigenvalue at Least -3

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Eigenvalues of Graphs

- All graphs in this talk are finite, undirected and simple.
- *Eigenvalues* of a graph G are the eigenvalues of its *adjacency matrix* $A(G)$:

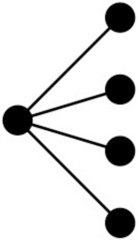
$$A(G)_{x,y} = \begin{cases} 1 & \text{if } x \text{ and } y \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{aligned} \lambda_{\min}(G) &= \textit{the smallest eigenvalue of } G \\ &= \text{the smallest eigenvalue of } A(G). \end{aligned}$$

We also denote by $\lambda_{\min}(M)$ the smallest eigenvalue of a real symmetric matrix M.

Claw $K_{1,k}$

$$A = \begin{pmatrix} 0 & 1 \cdots 1 \\ 1 & \\ \vdots & 0 \\ 1 & \end{pmatrix}$$

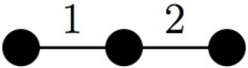


$$\det(xI - A) = x^{k-1}(x - \sqrt{k})(x + \sqrt{k}).$$

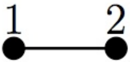
$\lambda_{\min}(k\text{-claw}) = -\sqrt{k}$:

- $\lambda_{\min}(G)$ can be arbitrarily small.
- Bounding $\lambda_{\min}$ from below $\implies \nexists k$-claw for large k.

The line graph $L(G)$



P_2



$L(P_2)$

Let $A = A(L(G))$, $C =$ edge-vertex incidence matrix

$$C = \text{edge} \begin{array}{c} \text{vertex} \\ \boxed{\begin{array}{c} 0\cdots010\cdots010\cdots0 \\ \\ \\ \end{array}} \end{array}$$

$$\lambda_{\min}(A) \geq \lambda_{\min}(A - CC^T) = \lambda_{\min}(-2I) = -2.$$

Maximal exceptional graphs

Theorem (Cameron–Goethals–Seidel–Shult, 1976)

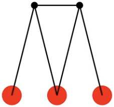
Every graph with smallest eigenvalue at least -2 is represented by a root system of type $\underbrace{A_n, D_n}_{\text{infinite}}$ or $\underbrace{E_8}_{\text{finite}}$.

Theorem (Cvetković–Rowlinson–Simić, 2002)

Every graph with smallest eigenvalue at least -2 is a *generalized line graph* or contained in one of the 473 maximal graphs represented by the root system E_8.

Hoffman's idea

$L(G)$ from G	Hoffman graph
$L(G)$	slim vertices
G	fat vertices



$$\begin{array}{cc} \text{slim} & \text{fat} \\ \end{array}$$
$$\begin{pmatrix} A & C \\ C^T & 0 \end{pmatrix}$$

$$\lambda_{\min}(A) \geq \lambda_{\min}(A - CC^T).$$

Advantage of considering $A - CC^T$ over A is that $A - CC^T$ is often a diagonal join of smaller matrices even if A is the adjacency matrix of a connected graph.

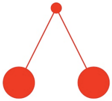
Woo and Neumaier (1995)

Definition

A (fat) *Hoffman graph* H is a graph (V, E) whose vertex set V consists of "slim" vertices and "fat" vertices, satisfying the following conditions:

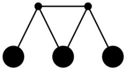
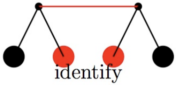
1. every slim vertex is adjacent to at least one fat vertex,
2. every fat vertex is adjacent to at least one slim vertex,
3. fat vertices are pairwise non-adjacent.

$$A(H) = \begin{array}{c} \begin{array}{cc} \text{slim} & \text{fat} \end{array} \\ \begin{pmatrix} A & C \\ C^T & 0 \end{pmatrix} \end{array} = \left(\begin{array}{c|cc} 0 & 1 & 1 \\ \hline 1 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right).$$

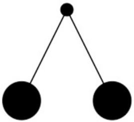


$$\lambda_{\min}(H) := \lambda_{\min}(A - CC^T).$$

Join and indecomposability

 is obtained by 

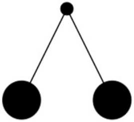
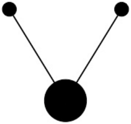
More generally, $H = H_1 \uplus H_2$ can be defined, for any graph G, its line graph is the slim part of a graph of the form $H_1 \uplus \cdots \uplus H_m$, where

$H_i \cong$ 

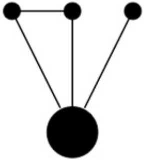
Note $\lambda_{\min}(H_i) = -2$, so

$$\begin{aligned} \lambda_{\min}(L(G)) &\geq \lambda_{\min}(H_1 \uplus \cdots \uplus H_m) \\ &= \min\{\lambda_{\min}(H_1), \cdots, \lambda_{\min}(H_m)\} = -2. \end{aligned}$$

Essentially, the only other indecomposable Hoffman graph with

$\lambda_{\min} \geq -2$ other than  is 

(This leads to the definition of a *generalized line graph*).

The next largest $\lambda_{\min}(H)$ is $-1-\sqrt{2}$: 

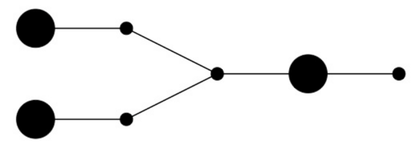
This gap between -2 and $-1-\sqrt{2}$ has implication in accumulation points of $\lambda_{\min}$ of ordinary graphs.

Theorem (Hoffman (1977))

If $\{G_n\}_{n=1}^{\infty}$ is a sequence of graphs with $d_{\min}(G_n) \to \infty$, $\lambda = \lim_{n\to\infty} \lambda_{\min}(G_n)$ exists and $\lambda < -2$, then $\lambda \le -1 - \sqrt{2}$.

Theorem (Woo and Neumaier (1995))

If $\{G_n\}_{n=1}^{\infty}$ is a sequence of graphs with $d_{\min}(G_n) \to \infty$, $\lambda = \lim_{n\to\infty} \lambda_{\min}(G_n)$ exists and $\lambda < -1 - \sqrt{2}$, then $\lambda \le \alpha$, where α is the smallest root of $x^3 + 2x^2 - 2x - 2$ and is the smallest eigenvalue of the Hoffman graph



$\alpha = -2.48119\cdots$

Conversely, $\lambda_{\min}$ of a Hoffman graph is a limit point of $\lambda_{\min}$ of ordinary graphs.

$\tau = \frac{1+\sqrt{5}}{2}$

Definition

A Hoffman graph H is λ-irreducible if $\lambda_{\min}(H) \geq \lambda$ and H cannot be embedded nontrivially to a Hoffman graph $H_1 \uplus H_2$ with $\lambda_{\min}(H_1), \lambda_{\min}(H_2) \geq \lambda$.

Theorem (M.–Sano–Taniguchi, arXiv:1111.7284v3)

There are exactly 37 $(-1-\tau)$-irreducible Hoffman graphs.

Note

$$-1-\tau = -2.618\cdots < \alpha < -1-\sqrt{2} < -2.$$

Edge-signed graphs

If a Hoffman graph H has adjacency matrix

$$\begin{array}{cc} \text{slim} & \text{fat} \\ \end{array}$$
$$\begin{pmatrix} A & C \\ C^T & 0 \end{pmatrix}$$

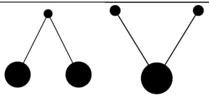
and all the off-diagonal entries of $A - CC^T$ are $0, \pm 1$, then one obtains an edge-signed graph S.

Theorem (Jang–Koolen–M.–Taniguchi, arXiv:1110.6821v1)

If $\lambda_{\min}(H) \geq -3$ and H has edge-signed graph S, then

- *the minus graph of S is the Dynkin graph $A_n, \tilde{A}_n, D_n, \tilde{D}_n$ (i.e., path plus 1 or 2 edges), or*
- *S is embeddable into the root system E_8 (hence finitely many possibilities).*

Summary

$\lambda_{\min}$	-1	-2	$-1-\sqrt{2}$	α
irreducible Hoffman graphs			Woo-Neumaier (4 graphs)	?
Hoffman type Theorem	Hoffman	Hoffman	Woo-Neumaier	?

$\lambda_{\min}$	$-1-\tau$	-3
irreducible Hoffman graphs	MST (37 graphs)	JKMT (signed graphs only)
Hoffman type Theorem	?	?