Complex Hadamard matrices contained in a Bose–Mesner algebra

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November 10, 2012
The 2nd 日本一台灣 Conference
on Combinatorics and its Applications
at 名古屋大學

Hadamard matrices and generalizations

- A Hadamard matrix of order n is an $n \times n$ matrix H with entries ± 1, satisfying $HH^T = nI$.
- A Butson-type Hadamard matrix of order n is an $n \times n$ matrix H with entries in $\{\zeta \in \mathbb{C} \mid \exists m \in \mathbb{N},\ \zeta^m = 1\}$, satisfying $HH^* = nI$.
- A complex Hadamard matrix of order n is an $n \times n$ matrix H with entries in $\{\zeta \in \mathbb{C} \mid |\zeta| = 1\}$, satisfying $HH^* = nI$.
- A type II (inverse orthogonal) matrix of order n is an $n \times n$ matrix H with entries in $\{\zeta \in \mathbb{C} \mid \zeta \neq 0\}$, satisfying $H{H^{(-)}}^T = nI$, where $H^{(-)}$ is the entrywise inverse.

$H = I + A_1 - A_2 \to H = I + x_1 A_1 + x_2 A_2$

Goethals–Seidel (1970):

Regular symmetric Hadamard matrices $\iff$ certain SRG.

$$H = I + A_1 - A_2$$

where

$$\begin{aligned} A_1 &= \text{adjacency matrix of a SRG } \Gamma, \\ A_2 &= \text{adjacency matrix of } \overline{\Gamma}. \end{aligned}$$

More generally, later we use "distance matrices":

$$(A_i)_{uv} = \begin{cases} 1 & \text{if } u, v \text{ are distance } i \text{ apart in } \Gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Strongly regular graphs and complex Hadamard matrices

Godsil–Chan (2010) classified type II (inverse orthogonal) matrices of the form:

$$H = I + x_1 A_1 + x_2 A_2$$

where $x_1, x_2 \in \mathbb{C}^\times$, and

$$\begin{aligned} A_1 &= \text{adjacency matrix of a SRG } \Gamma, \\ A_2 &= \text{adjacency matrix of } \overline{\Gamma}. \end{aligned}$$

Chan (`arXiv:1102.5601v1`) classified complex Hadamard matrices of the above form (i.e., $|x_1| = |x_2| = 1$).

$H = I + x_1A_1 + x_2A_2 + x_3A_3$

Chan (`arXiv:1102.5601v1`) considered:
Distance regular covers of complete graphs:

- a graph of diameter 3,
- "distance $=$ 0 or 3" is an equivalence relation,
- the distance matrices $A_0 = I, A_1, A_2, A_3$ span a 4-dim. vector space closed under multiplication.

Theorem (Chan)

For covers of complete graphs, if

$$H = I + x_1A_1 + x_2A_2 + x_3A_3$$

is a complex Hadamard matrix of order n, then $n \leq 16$.

$n = 15, L(O_3) =$ the line graph of the Petersen graph.

The line graph $L(O_3)$ of the Petersen graph O_3

$L(O_3)$ has 15 vertices. Its adjacency matrices satisfy

$$\begin{aligned} A_1^2 &= 4I + A_1 + A_2, \\ A_1 A_2 &= 2A_1 + 2A_2 + 4A_3, \\ A_1 A_3 &= A_2, \\ \cdots &= \cdots \end{aligned}$$

If $H = I + x_1 A_1 + x_2 A_2 + x_3 A_3$, then

$$HH^* = (I + x_1 A_1 + x_2 A_2 + x_3 A_3)(I + \overline{x_1} A_1 + \overline{x_2} A_2 + \overline{x_3} A_3)$$

$|x_i| = 1 \implies \overline{x_i} = x_i^{-1} \implies HH^* = 15I$ leads to polynomial equations for x_1, x_2, x_3.
Chan found only the solutions which are quadratic, but we found quartic ones $\implies$ generalized to an infinite family.

The Bose–Mesner algebra $\langle I, A_1, A_2, A_3\rangle$

A three-class association scheme consists of symmetric $(0, 1)$ matrices $A_0 = I, A_1, A_2, A_3$ satisfying

- $I + A_1 + A_2 + A_3 =$ all-one matrix,
- $\langle I, A_1, A_2, A_3\rangle$ is closed under multiplication (Bose–Mesner algebra)

Van Dam (1999) gives a list of order up to 100.
Question: Which three-class association scheme has a complex Hadamard matrix $H = I + x_1A_1 + x_2A_2 + x_3A_3$ in its Bose–Mesner algebra?
Chan: None for distance-regular covers of complete graphs except $n \leq 16$.

An infinite family

- q: a power of 2, $q \geq 4$,
- $\Omega = \mathrm{PG}(2, q)$: the projective plane over $\mathbb{F}_q$,
- $Q = \{[a_0, a_1, a_2] \in \Omega \mid a_0^2 + a_1 a_2 = 0\}$: quadric,
- $X = \{[a_0, a_1, a_2] \in \Omega \setminus Q \mid [a_0, a_1, a_2] \neq [1, 0, 0]\}$,
- $|X| = q^2 - 1$.

$$(A_i)_{xy} = \begin{cases} 1 & i = 1,\ |(x + y) \cap Q| = 2, \\ 1 & i = 2,\ |(x + y) \cap Q| = 0, \\ 1 & i = 3,\ |(x + y) \cap Q| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\langle I, A_1, A_2, A_3 \rangle$ is the Bose–Mesner algebra of a three-class association scheme, which is $L(O_3)$ when $q = 4$.

$\mathrm{PG}(2,q)$ (q: even), $H = I + x_1A_1 + x_2A_2 + x_3A_3$

$$s = \sqrt{(q-1)(17q-1)} > 0,$$

$$r_1 = \frac{-(q-1)(q-2) + (q+2)s}{2q(q+1)}.$$

Then $0 < r_1 < 2$. Let x_1 be one of the roots of

$$x_1^2 - r_1x_1 + 1 = 0.$$

Since the discriminant $r_1^2 - 4 < 0$, x_1 is imaginary, and $|x_1| = 1$.
Defining quartic imaginary numbers $x_2, x_3 \in \mathbb{Q}[x_1]$ with $|x_2| = |x_3| = 1$ suitably, one obtains a complex Hadamard matrix $H = I + x_1A_1 + x_2A_2 + x_3A_3$ of order $q^2 - 1$.

$H = I + x_1A_1 + x_2A_2 + x_3A_3$

$$x_2 = \frac{r_1x_1 - 2}{r_{12}x_1 - r_2},$$
$$x_3 = \frac{r_1x_1 - 2}{r_{13}x_1 - r_3},$$

where

$$r_2 = \frac{(q-1)(q+2) - (q-2)s}{2q(q-3)},$$
$$r_3 = \frac{(5q^2 - 2q - 19) - (q-1)s}{2(q+1)(q-3)},$$
$$r_{12} = \frac{-2(q^4 - 2q^3 - 4q^2 + 10q - 1) + 2(q-1)s}{q^2(q+1)(q-3)},$$
$$r_{13} = \frac{-(q+2)(q-1) + (q-2)s}{2q(q-3)}.$$