Complex Hadamard matrices contained in a Bose–Mesner algebra

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In this talk, a Hadamard matrix will mean a complex Hadamard matrix.

Circulant Hadamard matrix:

$$H = \begin{bmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \\ & \ddots & \ddots & \ddots \\ a_1 & & & a_0 \end{bmatrix}$$

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$$\zeta = \exp \frac{2\pi\sqrt{-1}}{n}, \quad F = \frac{1}{\sqrt{n}}(\zeta^{ij}) : \text{unitary},$$

$$F^* A F = D = \operatorname{diag}(\zeta^j; j = 0, 1, \ldots, n-1).$$

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Unifying principle: association schemes.
(strongly regular graphs is a special case)

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Godsil–Chan (2010): strongly regular graphs.

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where $a_1, a_2 \in \mathbb{C}^\times$, and

$$\begin{aligned} A_1 &= \text{adjacency matrix of a SRG } \Gamma, \\ A_2 &= \text{adjacency matrix of } \overline{\Gamma}. \end{aligned}$$

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$$H = \sum_{i=0}^{n-1} a_i A^i, \quad F^* A F = D$$

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Circulant Hadamard matrix. E_{jj}:matrix unit with (j,j) entry 1

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Construction analogous to circulant Hadamard matrix works if

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- F:unitary, $F^* A_i F = D_i$:diagonal
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- $A_0 = I, A_1, \ldots, A_d$:$(0,1)$-matrices, $\sum_i A_i = J$,
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Clearly $d \leq d'$. Equality holds iff

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Symmetric $\iff$ $A_i^\top = A_i$ $(\forall i)$ $\implies$ $p_{ji} \in \mathbb{R}$.

The Bose–Mesner algebra of a symmetric association scheme

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Indeed, image of f=zeros of g.

$$\begin{aligned} f : (\mathbb{C}^{\times})^4 &\longrightarrow \mathbb{C}^6, \\ (x_1, x_2, x_3, x_4) &\longmapsto \left(\tfrac{x_i}{x_j} + \tfrac{x_j}{x_i}\right)_{1 \leq i < j \leq 4} \end{aligned}$$

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(and similar polynomials obtained by permuting indices)

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$-2 < \alpha_{0i} < 2 \implies |a_i| = 1$.

- q: a power of 2, $q \geq 4$,
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- $X = \{[a_0, a_1, a_2] \in \Omega \setminus Q \mid [a_0, a_1, a_2] \neq [1, 0, 0]\}$,
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$$(A_i)_{xy} = \begin{cases} 1 & i = 1,\ |(x + y) \cap Q| = 2, \\ 1 & i = 2,\ |(x + y) \cap Q| = 0, \\ 1 & i = 3,\ |(x + y) \cap Q| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

$\exists$ a complex Hadamard matrix in its Bose–Mesner algebra.

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- $|X| = q^2 - 1 \to 15$.

$$(A_i)_{xy} = \begin{cases} 1 & i = 1,\ |(x+y) \cap Q| = 2, \\ 1 & i = 2,\ |(x+y) \cap Q| = 0, \\ 1 & i = 3,\ |(x+y) \cap Q| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

$\exists$ a complex Hadamard matrix in $L(O_3)$.