Codes Generated by Designs, and Designs Supported by Codes Part I

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May, 2013
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t-(v, k, λ) designs

Definition

A t-(v, k, λ) design is a pair $(\mathcal{P}, \mathcal{B})$, where

- $\mathcal{P}$: a finite set of "points",
- $\mathcal{B}$: a collection of k-subsets of $\mathcal{P}$, a member of which is called a "block,"
- $\forall T \subset \mathcal{P}$ with $|T| = t$, there are exactly λ members $B \in \mathcal{B}$ such that $T \subset B$.

Examples:

- 2-$(v, 3, 1)$ design $=$ Steiner triple system
- 2-$(q^2, q, 1)$ design $=$ affine plane of order q

$$t\text{-design} \implies (t-1)\text{-design}$$

More precisely,. . .

Intersection numbers

$(\mathcal{P}, \mathcal{B})$: t-(v, k, λ) design. Write $\lambda = \lambda_t$,

$$\lambda_{t-1} = |\{B \in \mathcal{B} \mid T' \subset B\}|,$$

where $T' \subset \mathcal{P}$, $|T'| = t - 1$. Then

$$\begin{aligned}
\lambda_{t-1}(k - t + 1) &= \sum_{\substack{B \in \mathcal{B} \\ T' \subset B}} |B \setminus T'| \\
&= |\{(B, x) \in \mathcal{B} \mid T' \cup \{x\} \subset B,\ x \in \mathcal{P} \setminus T'\}| \\
&= \sum_{x \in \mathcal{P} \setminus T'} |\{B \in \mathcal{B} \mid T' \cup \{x\} \subset B\}| \\
&= \sum_{x \in \mathcal{P} \setminus T'} \lambda_t \\
&= \lambda_t(v - t + 1).
\end{aligned}$$

$(\mathcal{P}, \mathcal{B})$: t-(v, k, λ) design

Then $(\mathcal{P}, \mathcal{B})$: $(t-1)$-(v, k, λ_{t-1}) design, where

$$\lambda_{t-1} = \lambda_t \frac{v-t+1}{k-t+1}.$$

For example,

$$\begin{aligned} 5\text{-}(24, 8, 1) &\implies 4\text{-}(24, 8, 5) \\ &\implies 3\text{-}(24, 8, 21) \\ &\implies 2\text{-}(24, 8, 77) \\ &\implies 1\text{-}(24, 8, 253) \\ &\implies 0\text{-}(24, 8, 759) \\ &\iff |\mathcal{B}| = 759. \end{aligned}$$

$(\mathcal{P}, \mathcal{B})$: t-(v, k, λ) design

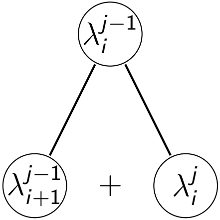
Let $I \subset \mathcal{P}$, $J \subset \mathcal{P}$, $|I| = i$, $|J| = j$, $I \cap J = \emptyset$, $i + j \leq t$.
Define

$$\lambda_i^j = |\{B \in \mathcal{B} \mid I \subset B,\ B \cap J = \emptyset\}|.$$

In particular,

$$\lambda_i^0 = \lambda_i \quad (0 \leq i \leq t).$$

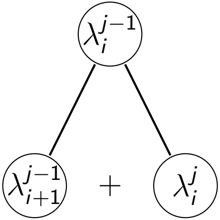
$$\lambda_i^{j-1} = \lambda_{i+1}^{j-1} + \lambda_i^j.$$



$$\begin{array}{c}
\lambda_0^0 \\
\lambda_1^0\ \lambda_0^1 \\
\lambda_2^0\ \lambda_1^1\ \lambda_0^2 \\
\lambda_3^0\ \lambda_2^1\ \lambda_1^2\ \lambda_0^3 \\
\lambda_4^0\ \lambda_3^1\ \lambda_2^2\ \lambda_1^3\ \lambda_0^4 \\
\lambda_5^0\ \lambda_4^1\ \lambda_3^2\ \lambda_2^3\ \lambda_1^4\ \lambda_0^5
\end{array}$$

5-(24, 8, 1) design, $\lambda_i^{j-1} = \lambda_{i+1}^{j-1} + \lambda_i^j$

759
253 506
77 176 330
21 56 120 210
5 16 40 80 130
1 4 12 28 52 78



Next row?
λ_6^0, λ_5^1, λ_4^2, ...

$$\lambda_6^0(I) = |\{B \in \mathcal{B} \mid I \subset B\}| = 1 \text{ or } 0$$

depending on the choice of $I \subset \mathcal{P}$ with $|I| = 6$.
Choose I in such a way that $\lambda_6^0(I) = 1$.

5-(24, 8, 1) design, $I \subset \mathcal{P}$, $|I| = 6$, $I \subset \exists B \in \mathcal{B}$

$$\lambda^{j}_{6-j} = |\{B \in \mathcal{B} \mid I \setminus J \subset B,\ B \cap J = \emptyset\}| \quad \text{where } J \subset I,\ J = j.$$

$$\lambda^{j}_{5-j} = \lambda^{j}_{6-j} + \lambda^{j+1}_{5-j}$$

giving

759
253 506
77 176 330
21 56 120 210
5 16 40 80 130
1 4 12 28 52 78
1 0 4 8 20 32 46

Similarly, taking $I \subset \mathcal{P}$, $|I| = 7$ appropriately, we obtain λ^{j}_{7-j}.
Finally taking $I \in \mathcal{B}$, we obtain λ^{j}_{8-j}.

5-(24, 8, 1) design

$$
\begin{array}{c}
759 \\
253 \quad 506 \\
77 \quad 176 \quad 330 \\
21 \quad 56 \quad 120 \quad 210 \\
5 \quad 16 \quad 40 \quad 80 \quad 130 \\
1 \quad 4 \quad 12 \quad 28 \quad 52 \quad 78 \\
1 \quad 0 \quad 4 \quad 8 \quad 20 \quad 32 \quad 46 \\
1 \quad 0 \quad 0 \quad 4 \quad 4 \quad 16 \quad 16 \quad 30 \\
1 \quad 0 \quad 0 \quad 0 \quad 4 \quad 0 \quad 16 \quad 0 \quad 30
\end{array}
$$

The last row implies

$$B, B' \in \mathcal{P},\ B \neq B' \implies |B \cap B'| \in \{4, 2, 0\}.$$

The 5-(24, 8, 1) design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

1	2	3	4	5	6	7	8																
1	2	3	4					9	10	11	12												
1	2	3		5				9				13	14	15									
1	2		4	5				9							16	17	18						
1		3	4	5				9										19	20	21			
	2	3	4	5				9													22	23	24
1	2	3			6			9							16			19			22		
1	2		4		6			9				13							20			23	
1		3	4		6			9					14			17							24
1	2			5	6			9	10											21			24
1		3		5	6			9		11							18					23	

Do we have to find 759 blocks one by one?
No, 12 blocks are sufficient (so one more needed).

Todd's lemma

Let $(\mathcal{P}, \mathcal{B})$ be a 5-$(24, 8, 1)$ design. Then

$$B, B' \in \mathcal{B},\ |B \cap B'| = 4 \implies B \triangle B' \in \mathcal{B}.$$

Proof by contradiction:

```
1 2 3 4 5 6 7 8
1 2 3 4         9 10 11 12
        5 6 7 8 9 10       13 14
        5 6 7 8      11 12       15 16
* * * * 5 6 7   9    11
```

Here "$****$" must be odd and even simultaneously.

Consequence of Todd's lemma

1	2	3	4	5	6	7	8																
1	2	3	4					9	10	11	12												
1	2	3		5				9				13	14	15									
1	2		4	5				9							16	17	18						
1		3	4	5				9										19	20	21			
	2	3	4	5				9													22	23	24
1	2	3			6			9							16			19			22		
1	2		4		6			9				13							20			23	
1		3	4		6			9					14			17							24
1	2			5	6			9	10											21			24
1		3		5	6			9		11							18					23	

By Todd's lemma

$$((B_1 \triangle B_4) \triangle B_7) \triangle (B_5 \triangle B_6) = \{7, 8, 17, 18, 20, 21, 23, 24\} \in \mathcal{B}.$$

Binary codes

A (linear) binary code of length v is a subspace of the vector space $\mathbb{F}_2^v$. If C is a binary code and $\dim C = k$, we say C is an binary $[v, k]$ code.
The dual code of a binary code C is defined as

$$C^\perp = \{x \in \mathbb{F}_2^v \mid x \cdot y = 0 \ (\forall y \in C)\}.$$

where

$$x \cdot y = \sum_{i=1}^{v} x_i y_i.$$

Then

$$\dim C^\perp = v - \dim C.$$

The code C is said to be self-orthogonal if $C \subset C^\perp$ and self-dual if $C = C^\perp$.

Generator matrix of a code

If a binary code C is generated by row vectors $x^{(1)}, \dots, x^{(b)}$, then the matrix

$$\begin{bmatrix} x^{(1)} \\ \vdots \\ x^{(b)} \end{bmatrix}$$

is called a generator matrix of C. This means

$$C = \{\sum_{i=1}^{b} \epsilon_i x^{(i)} \mid \epsilon_1, \dots, \epsilon_b \in \mathbb{F}_2\} \subset \mathbb{F}_2^v.$$

Incidence matrix of a design

If $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is a t-(v, k, λ) design, the incidence matrix $M(\mathcal{D})$ of $\mathcal{D}$ is $|\mathcal{B}| \times |\mathcal{P}|$ matrix whose rows and columns are indexed by $\mathcal{B}$ and $\mathcal{P}$, respectively, such that its (B, p) entry is 1 if $p \in B$, 0 otherwise. In other words, the row vectors of $M(\mathcal{D})$ are the characteristic vectors of blocks:

$$M(\mathcal{D}) = \begin{bmatrix} x^{(B_1)} \\ \vdots \\ x^{(B_b)} \end{bmatrix} \quad : b \times v \text{ matrix},$$

where $\mathcal{B} = \{B_1, \dots, B_b\}$, and $x^{(B)} \in \mathbb{F}_2^v$ denotes the characteristic vector of B.
The binary code of the design $\mathcal{D}$ is the binary code of length v having $M(\mathcal{D})$ as a generator matrix.

$\dim C \le 12$ for 5-$(24, 8, 1)$ design

Recall that in a 5-$(24, 8, 1)$ design $(\mathcal{P}, \mathcal{B})$,

$$|B \cap B'| \in \{8, 4, 2, 0\} \quad (\forall B, B' \in \mathcal{B}).$$

The binary code C of a 5-$(24, 8, 1)$ design is self-orthogonal. Indeed, the incidence matrix has row vectors $x^{(B)}$ $(B \in \mathcal{B})$, the characteristic vector of the block B. Then

$$x^{(B)} \cdot x^{(B')} = |B \cap B'| \bmod 2 = (8 \text{ or } 4 \text{ or } 2 \text{ or } 0) \bmod 2 = 0.$$

Thus $C \subset C^{\perp}$, hence

$$\dim C \le \frac{1}{2}(\dim C + \dim C^{\perp}) \le \frac{24}{2} = 12.$$

One more block for 5-$(24, 8, 1)$ design

1	2	3	4	5	6	7	8																
1	2	3	4					9	10	11	12												
1	2	3		5				9				13	14	15									
1	2		4	5				9							16	17	18						
1		3	4	5				9										19	20	21			
	2	3	4	5				9													22	23	24
1	2	3			6			9							16			19			22		
1	2		4		6			9				13							20			23	
1		3	4		6			9					14			17							24
1	2			5	6			9	10											21			24
1		3		5	6			9		11							18					23	

The above 11 blocks generate a 11-dimensional code C_0. Note the transposition (7 8) leaves C_0 invariant. We know from Todd's lemma $B_0 = \{7, 8, 17, 18, 20, 21, 23, 24\} \in \mathcal{B}$ (but $x^{(B_0)} \in C_0$).
Consider the block containing $\{1, 2, 3, 8, 9\}$. There are two choices: $B = \{1, 2, 3, 8, 9, 17, 21, 23\}$ and $B' = \{1, 2, 3, 8, 9, 18, 20, 24\}$.

One more block for 5-$(24, 8, 1)$ design

We know

$$B_0 = \{7, 8, 17, 18, 20, 21, 23, 24\} \in \mathcal{B}, \quad x^{(B_0)} \in C_0 = C_0^{(7\ 8)}.$$

We have either

$$B = \{1, 2, 3, 8, 9, 17, 21, 23\} \in \mathcal{B} \text{ or}$$
$$B' = \{1, 2, 3, 8, 9, 18, 20, 24\} \in \mathcal{B}.$$

But $B'^{(7\ 8)} = B \triangle B_0$, so

$$\langle C_0, x^{(B')} \rangle^{(7\ 8)} = \langle C_0, x^{(B)} + x^{(B_0)} \rangle = \langle C_0, x^{(B)} \rangle.$$

Therefore, the code generated by the design is unique up to isomorphism. This self-dual ($C = C^{\perp}$) code is known as the extended binary Golay code. Next we show that the code determines the design uniquely.

Weight

For $x \in \mathbb{F}_2^v$, we write

$$\begin{aligned} \text{supp}(x) &= \{i \mid 1 \le i \le v,\ x_i \ne 0\}, \\ \text{wt}(x) &= |\,\text{supp}(x)|. \end{aligned}$$

For a binary code C, its minimum weight is

$$\min\{\text{wt}(x) \mid 0 \ne x \in C\}.$$

If an $[v, k]$ code C has minimum weight d, we call C an $[v, k, d]$ code.

Mendelsohn equations for t-(v, k, λ) design $(\mathcal{P}, \mathcal{B})$

For $S \subset \mathcal{P}$, let

$$n_i(S) = |\{B \in \mathcal{B} \mid i = |B \cap S|\}|.$$

Then

$$\sum_{i \geq 0} \binom{i}{j} n_i(S) = \lambda_j \binom{|S|}{j} \quad (0 \leq j \leq t).$$

Proof: Count

$$\{(J, B) \mid J \subset S \cap B,\ |J| = j\}$$

in two ways.

$n_i(S) = |\{B \in \mathcal{B} \mid i = |B \cap S|\}|$

Let C be the binary code of the design $(\mathcal{P}, \mathcal{B})$.
Write $n_i(\mathrm{supp}(v)) = n_i(v)$ for $v \in \mathbb{F}_2^v$.

$$\sum_{i \geq 0} \binom{i}{j} n_i(v) = \lambda_j \binom{\mathrm{wt}(v)}{j} \quad (0 \leq j \leq t).$$

If $v \in C^\perp$, then $|B \cap \mathrm{supp}(v)|$ is even, so

$$n_i(v) = |\{B \in \mathcal{B} \mid i = |B \cap \mathrm{supp}(v)|\}| = 0 \quad \text{for } i \text{ odd}.$$

Thus

$$\sum_{\substack{0 \leq i \leq \mathrm{wt}(v) \\ i:\ \text{even}}} \binom{i}{j} n_i(v) = \lambda_j \binom{\mathrm{wt}(v)}{j} \quad (0 \leq j \leq t).$$

$(\mathcal{P}, \mathcal{B})$: 5-$(24, 8, 1)$ design

$$\sum_{\substack{0 \le i \le \mathrm{wt}(v) \\ i:\ \text{even}}} \binom{i}{j} n_i(v) = \lambda_j \binom{\mathrm{wt}(v)}{j} \quad (0 \le j \le 5).$$

Taking $v \in C^\perp$ with $0 < \mathrm{wt}(v) < 8$ gives no solution. This means that $C^\perp$ has minimum weight 8.
Take $v \in C = C^\perp$ with $\mathrm{wt}(v) = 8$. Then there are six equations for five unknowns n_0, n_2, n_4, n_6, n_8. The unique solution is

$$(n_0, n_2, n_4, n_6, n_8) = (30, 448, 280, 0, 1).$$

This implies $\mathrm{supp}(v) \in \mathcal{B}$. Thus

$$\mathcal{B} = \{\mathrm{supp}(x) \mid x \in C,\ \mathrm{wt}(x) = 8\}.$$

Now the uniqueness of the design follows from that of C.

C: the binary code of a 5-$(24, 8, 1)$ design

For $v \in C^\perp$,

$$\sum_{\substack{0 \le i \le \mathrm{wt}(v) \\ i:\ \text{even}}} \binom{i}{j} n_i(v) = \lambda_j \binom{\mathrm{wt}(v)}{j} \quad (0 \le j \le 5).$$

Taking $\mathrm{wt}(v) = 10$ gives a unique solution which is not integral. This means that $C^\perp$ has no vectors of weight 10.

weight	0	8	12	16	24
# vectors	1	759	2576	759	1

- C is generated by vectors of weight 8 $\implies$ $C^\perp$ contains the all-one vector $\implies$ the weight distribution of $C^\perp$ is symmetric.
- $C^\perp$ contains only vectors of weight divisible by 4 (such a code is called doubly even) $\implies$ $C^\perp \subset (C^\perp)^\perp = C$, forcing $C = C^\perp$.

Summary

$\mathcal{D}$: 5-$(24, 8, 1)$ design (Witt system).

- The binary code C of $\mathcal{D}$ is a doubly even self-dual $[24, 12, 8]$ code.
- The binary code C of $\mathcal{D}$ is unique up to isomorphism.
- $\{\mathrm{supp}(x) \mid x \in C,\ \mathrm{wt}(x) = 8\} = \mathcal{B}$.
- There is a unique 5-$(24, 8, 1)$ design up to isomorphism.

The Assmus–Mattson theorem implies that every binary doubly even self-dual $[24, 12, 8]$ code coincides with the binary code of a 5-$(24, 8, 1)$ design, and hence such a code (the extended binary Golay code) is also unique.

The next two lectures will cover

- proof of the Assmus–Mattson Theorem
- characterization of the (binary) Hadamard matrix contained in the set of vectors of weight 12 in the extended binary Golay $[24, 12, 8]$ code.