Codes Generated by Designs, and Designs Supported by Codes Part I

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Graphs, Codes, and Designs
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t-(v, k, λ) designs

Definition

A t-(v, k, λ) design is a pair $(\mathcal{P}, \mathcal{B})$, where

- $\mathcal{P}$: a finite set of v "points",
- $\mathcal{B}$: a collection of k-subsets of $\mathcal{P}$, a member of which is called a "block,"
- $\forall T \subset \mathcal{P}$ with $|T| = t$, there are exactly λ members $B \in \mathcal{B}$ such that $T \subset B$.

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$$t\text{-design} \implies (t-1)\text{-design}$$

More precisely,...

Intersection numbers

$(\mathcal{P}, \mathcal{B})$: t-(v, k, λ) design. Write $\lambda = \lambda_t$,

$$\lambda_{t-1} = |\{B \in \mathcal{B} \mid T' \subset B\}|,$$

where $T' \subset \mathcal{P}$, $|T'| = t - 1$. Then

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$$\begin{aligned}
\lambda_{t-1}(k - t + 1) &= \sum_{\substack{B \in \mathcal{B} \\ T' \subset B}} |B \setminus T'| \\
&= |\{(B, x) \mid B \in \mathcal{B},\ T' \cup \{x\} \subset B,\ x \in \mathcal{P} \setminus T'\}| \\
&= \sum_{x \in \mathcal{P} \setminus T'} |\{B \in \mathcal{B} \mid T' \cup \{x\} \subset B\}| \\
&= \sum_{x \in \mathcal{P} \setminus T'} \lambda_t \\
&= \lambda_t(v - t + 1).
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$(\mathcal{P}, \mathcal{B})$: t-(v, k, λ) design

Then $(\mathcal{P}, \mathcal{B})$: $(t-1)$-(v, k, λ_{t-1}) design, where

$$\lambda_{t-1} = \lambda_t \frac{v-t+1}{k-t+1}$$

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Then $(\mathcal{P}, \mathcal{B})$: $(t-1)$-(v, k, λ_{t-1}) design, where

$$\lambda_{t-1} = \lambda_t \frac{v-t+1}{k-t+1} = 1 \cdot \frac{24-5+1}{8-5+1} = \frac{20}{4} = 5$$

For example,

$$5\text{-}(24, 8, 1) \implies 4\text{-}(24, 8, 5)$$

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For example,

$$\begin{aligned} 5\text{-}(24, 8, 1) &\implies 4\text{-}(24, 8, 5) \\ &\implies 3\text{-}(24, 8, 21) \\ &\implies 2\text{-}(24, 8, 77) \end{aligned}$$

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$(\mathcal{P}, \mathcal{B})$: t-(v, k, λ) design

Let $I \subset \mathcal{P}$, $J \subset \mathcal{P}$, $|I| = i$, $|J| = j$, $I \cap J = \emptyset$, $i + j \leq t$.

$(\mathcal{P}, \mathcal{B})$: t-(v, k, λ) design

Let $I \subset \mathcal{P}$, $J \subset \mathcal{P}$, $|I| = i$, $|J| = j$, $I \cap J = \emptyset$, $i + j \leq t$.
Define

$$\lambda_i^j = |\{B \in \mathcal{B} \mid I \subset B,\ B \cap J = \emptyset\}|.$$

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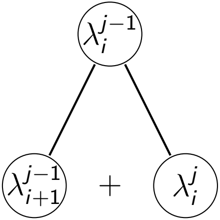

$$\begin{array}{c} \lambda_0^0 \\ \lambda_1^0\ \lambda_0^1 \\ \lambda_2^0\ \lambda_1^1\ \lambda_0^2 \\ \lambda_3^0\ \lambda_2^1\ \lambda_1^2\ \lambda_0^3 \\ \lambda_4^0\ \lambda_3^1\ \lambda_2^2\ \lambda_1^3\ \lambda_0^4 \\ \lambda_5^0\ \lambda_4^1\ \lambda_3^2\ \lambda_2^3\ \lambda_1^4\ \lambda_0^5 \end{array}$$

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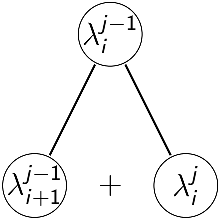
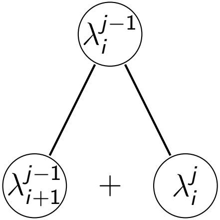


$$\begin{array}{c}
\lambda_0^0 \\
\lambda_1^0\ \lambda_0^1 \\
\lambda_2^0\ \lambda_1^1\ \lambda_0^2 \\
\lambda_3^0\ \lambda_2^1\ \lambda_1^2\ \lambda_0^3 \\
\lambda_4^0\ \lambda_3^1\ \lambda_2^2\ \lambda_1^3\ \lambda_0^4 \\
\lambda_5^0\ \lambda_4^1\ \lambda_3^2\ \lambda_2^3\ \lambda_1^4\ \lambda_0^5
\end{array}$$

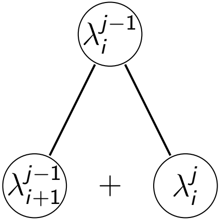
5-$(24, 8, 1)$ design, $\lambda_i^{j-1} = \lambda_{i+1}^{j-1} + \lambda_i^j$

759
253
77
21
5
1



5-$(24, 8, 1)$ design, $\lambda_i^{j-1} = \lambda_{i+1}^{j-1} + \lambda_i^j$

					759	
				253		506
			77		176	
		21		56		
	5		16			
1		4				



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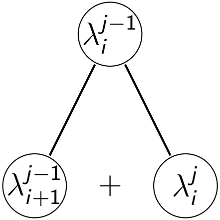
					759
				253	506
			77	176	330
		21	56	120	
	5	16	40		
1	4	12			



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759
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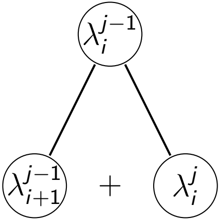
77 176 330
21 56 120 210
5 16 40 80
1 4 12 28



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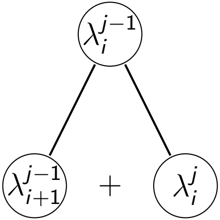
5 16 40 80 130
1 4 12 28 52



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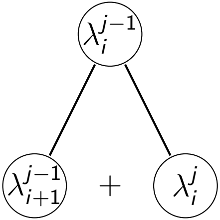

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759

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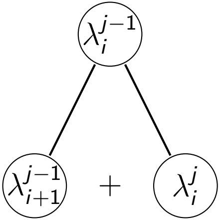
Next row? λ_6^0, λ_5^1, λ_4^2, ...



5-$(24, 8, 1)$ design, $\lambda_i^{j-1} = \lambda_{i+1}^{j-1} + \lambda_i^j$

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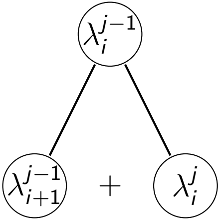


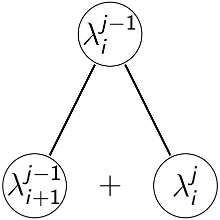
Next row? λ_6^0, λ_5^1, λ_4^2, ...

$$\lambda_6^0(I) = |\{B \in \mathcal{B} \mid I \subset B\}| = 1 \text{ or } 0$$

depending on the choice of $I \subset \mathcal{P}$ with $|I| = 6$.

5-(24, 8, 1) design, $\lambda_i^{j-1} = \lambda_{i+1}^{j-1} + \lambda_i^j$

$$\begin{array}{ccccccccccc}
 & & & & & 759 & & & & & \\
 & & & & 253 & & 506 & & & & \\
 & & & 77 & & 176 & & 330 & & & \\
 & & 21 & & 56 & & 120 & & 210 & & \\
 & 5 & & 16 & & 40 & & 80 & & 130 & \\
1 & & 4 & & 12 & & 28 & & 52 & & 78
\end{array}$$



Next row? λ_6^0, λ_5^1, λ_4^2, ...

$$\lambda_6^0(I) = |\{B \in \mathcal{B} \mid I \subset B\}| = 1 \text{ or } 0$$

depending on the choice of $I \subset \mathcal{P}$ with $|I| = 6$.
Choose I in such a way that $\lambda_6^0(I) = 1$.

5-(24, 8, 1) design, $I \subset \mathcal{P}$, $|I| = 6$, $I \subset \exists B \in \mathcal{B}$

$$\lambda^{j}_{6-j} = |\{B \in \mathcal{B} \mid I \setminus J \subset B,\ B \cap J = \emptyset\}| \quad \text{where } J \subset I,\ J = j.$$

$$\lambda^{j}_{5-j} = \lambda^{j}_{6-j} + \lambda^{j+1}_{5-j}$$

giving

759						
253	506					
77	176	330				
21	56	120	210			
5	16	40	80	130		
1	4	12	28	52	78	
1	0	4	8	20	32	46

Similarly, taking $I \subset \mathcal{P}$, $|I| = 7$ appropriately, we obtain λ^{j}_{7-j}.
Finally taking $I \in \mathcal{B}$, we obtain λ^{j}_{8-j}.

5-$(24, 8, 1)$ design

```
                    759
                 253   506
              77    176   330
           21    56    120   210
         5    16    40    80    130
       1    4    12    28    52    78
     1    0    4     8     20    32    46
   1    0    0    4     4     16    16    30
1    0    0    0    4     0     16    0     30
```

The last row:

$$\lambda^{j}_{8-j} = |\{B \in \mathcal{B} \mid I \setminus J \subset B,\ B \cap J = \emptyset\}| \quad \text{where } J \subset I \in \mathcal{B},\ J = j.$$

$$B, B' \in \mathcal{P},\ B \neq B' \implies |B \cap B'| \in \{4, 2, 0\}.$$

The 5-$(24, 8, 1)$ design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1 1 2 3 4 5 6 7 8

The 5-$(24, 8, 1)$ design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8	
B_2	1	2	3	4	5	6	7	8	9

The 5-$(24, 8, 1)$ design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1 1 2 3 4 5 6 7 8
B_2 1 2 3 4 5 6 7 8 9 10 11 12

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B_2	1	2	3	4					9	10	11	12

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$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8				
B_2	1	2	3	4					9	10	11	12
B_3	1	2	3		5				9			

The 5-(24, 8, 1) design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8				
B_2	1	2	3	4					9	10	11	12
B_3	1	2	3	4	5	6	7	8	9	10	11	12

The 5-(24, 8, 1) design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

B_1	1	2	3	4	5	6	7	8							
B_2	1	2	3	4					9	10	11	12			
B_3	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

The 5-$(24, 8, 1)$ design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \ldots, 24\}$. We may take $\mathcal{B}$ as:

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B_3	1	2	3		5				9				13	14	15

The 5-$(24, 8, 1)$ design, $|B \cap B'| \in \{4, 2, 0\}$

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Do we have to find 759 blocks one by one?

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No, only 12 blocks are sufficient.

Binary codes

A (linear) binary code of length v is a subspace of the vector space $\mathbb{F}_2^v$. If C is a binary code and $\dim C = k$, we say C is an binary $[v, k]$ code.

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$$C^\perp = \{x \in \mathbb{F}_2^v \mid x \cdot y = 0 \ (\forall y \in C)\}.$$

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Then

$$\dim C^\perp = v - \dim C.$$

The code C is said to be self-orthogonal if $C \subset C^\perp$ and self-dual if $C = C^\perp$.

Generator matrix of a code

If a binary code C is generated by row vectors $g^{(1)}, \dots, g^{(b)}$, then the matrix

$$\begin{bmatrix} g^{(1)} \\ \vdots \\ g^{(b)} \end{bmatrix}$$

is called a generator matrix of C. This means

$$C = \{\sum_{i=1}^{b} \epsilon_i g^{(i)} \mid \epsilon_1, \dots, \epsilon_b \in \mathbb{F}_2\} \subset \mathbb{F}_2^v.$$

Incidence matrix of a design

If $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is a t-(v, k, λ) design, the incidence matrix $M(\mathcal{D})$ of $\mathcal{D}$ is $|\mathcal{B}| \times |\mathcal{P}|$ matrix whose rows and columns are indexed by $\mathcal{B}$ and $\mathcal{P}$, respectively, such that its (B, p) entry is 1 if $p \in B$, 0 otherwise.

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$$M(\mathcal{D}) = \begin{bmatrix} x^{(B_1)} \\ \vdots \\ x^{(B_b)} \end{bmatrix} \quad : b \times v \text{ matrix},$$

where $\mathcal{B} = \{B_1, \ldots, B_b\}$, and $x^{(B)} \in \mathbb{F}_2^v$ denotes the characteristic vector of B.

Incidence matrix of a design

If $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is a t-(v, k, λ) design, the incidence matrix $M(\mathcal{D})$ of $\mathcal{D}$ is $|\mathcal{B}| \times |\mathcal{P}|$ matrix whose rows and columns are indexed by $\mathcal{B}$ and $\mathcal{P}$, respectively, such that its (B, p) entry is 1 if $p \in B$, 0 otherwise. In other words, the row vectors of $M(\mathcal{D})$ are the characteristic vectors of blocks:

$$M(\mathcal{D}) = \begin{bmatrix} x^{(B_1)} \\ \vdots \\ x^{(B_b)} \end{bmatrix} \quad : b \times v \text{ matrix,}$$

where $\mathcal{B} = \{B_1, \dots, B_b\}$, and $x^{(B)} \in \mathbb{F}_2^v$ denotes the characteristic vector of B.

The binary code of the design $\mathcal{D}$ is the binary code of length v having $M(\mathcal{D})$ as a generator matrix.

dim $C \leq 12$ for 5-$(24, 8, 1)$ design

Recall that in a 5-$(24, 8, 1)$ design $(\mathcal{P}, \mathcal{B})$,

$$|B \cap B'| \in \{8, 4, 2, 0\} \quad (\forall B, B' \in \mathcal{B}).$$

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The binary code C of a 5-$(24, 8, 1)$ design is self-orthogonal. Indeed, the incidence matrix has row vectors $x^{(B)}$ $(B \in \mathcal{B})$, the characteristic vector of the block B. Then

$$x^{(B)} \cdot x^{(B')} = |B \cap B'| \bmod 2 = (8 \text{ or } 4 \text{ or } 2 \text{ or } 0) \bmod 2 = 0.$$

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Thus $C \subset C^{\perp}$, hence

$$\dim C \leq \frac{1}{2}(\dim C + \dim C^{\perp}) \leq \frac{24}{2} = 12.$$

Weight

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For $x \in \mathbb{F}_2^v$, we write

$$\begin{aligned} \mathrm{supp}(x) &= \{i \mid 1 \le i \le v,\ x_i \ne 0\}, \\ \mathrm{wt}(x) &= |\,\mathrm{supp}(x)|. \end{aligned}$$

For a binary code C, its minimum weight is

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If an $[v, k]$ code C has minimum weight d, we call C an $[v, k, d]$ code.

Mendelsohn equations for t-(v, k, λ) design $(\mathcal{P}, \mathcal{B})$

For $S \subset \mathcal{P}$, let

$$n_i(S) = |\{B \in \mathcal{B} \mid i = |B \cap S|\}|.$$

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$$\{(J, B) \mid J \subset S \cap B,\ |J| = j\}$$

in two ways.

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If $v \in C^\perp$, then $|B \cap \mathrm{supp}(v)|$ is even, so

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Thus

$$\sum_{\substack{0 \leq i \leq \mathrm{wt}(v) \\ i:\mathrm{even}}} \binom{i}{j} n_i(v) = \lambda_j \binom{\mathrm{wt}(v)}{j} \quad (0 \leq j \leq t).$$

$(\mathcal{P}, \mathcal{B})$: 5-$(24, 8, 1)$ design

$$\sum_{\substack{0 \le i \le \mathrm{wt}(v) \\ i:\ \text{even}}} \binom{i}{j} n_i(v) = \lambda_j \binom{\mathrm{wt}(v)}{j} \quad (0 \le j \le 5).$$

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Take $v \in C = C^\perp$ with $\mathrm{wt}(v) = 8$. Then there are six equations for five unknowns n_0, n_2, n_4, n_6, n_8. The unique solution is

$$(n_0, n_2, n_4, n_6, n_8) = (30, 448, 280, 0, 1).$$

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This implies $\mathrm{supp}(v) \in \mathcal{B}$. Thus

$$\mathcal{B} = \{\mathrm{supp}(x) \mid x \in C,\ \mathrm{wt}(x) = 8\}.$$

Now the uniqueness of the design follows from that of C.

C: the binary code of a 5-$(24, 8, 1)$ design

For $v \in C^{\perp}$,

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Taking $\mathrm{wt}(v) = 10$ gives a unique solution which is not integral. This means that $C^{\perp}$ has no vectors of weight 10.

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weight	0	8	12	16	24	total
# vectors	1	759	2576	759	1	2^{12}

- C is generated by vectors of weight 8 $\implies$ $C^{\perp}$ contains the all-one vector $\implies$ C contains the all-one vector $\implies$ the weight distribution of C is symmetric, no odd-weight vectors.

Summary

$\mathcal{D}$: 5-$(24, 8, 1)$ design (Witt system).

- The binary code C of $\mathcal{D}$ is a doubly even (all weights $\equiv 0 \pmod 4$) self-dual $[24, 12, 8]$ code.
- The binary code C of $\mathcal{D}$ is unique up to isomorphism.
- $\{\text{supp}(x) \mid x \in C, \ \text{wt}(x) = 8\} = \mathcal{B}$.
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The next two lectures will cover

- proof of the Assmus–Mattson Theorem
- characterization of the (binary) Hadamard matrix contained in the set of vectors of weight 12 in the extended binary Golay $[24, 12, 8]$ code.