Codes Generated by Designs, and Designs Supported by Codes Part II

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Contents

Summary of Part I

$\mathcal{D}$: 5-$(24, 8, 1)$ design (Witt system).

- The binary code C of $\mathcal{D}$ is a doubly even self-dual $[24, 12, 8]$ code.
- $\{\text{supp}(x) \mid x \in C,\ \text{wt}(x) = 8\} = \mathcal{B}$.
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Part II will cover

- proof of the Assmus–Mattson theorem
- other 5-designs obtained from doubly even self-dual codes

The Assmus–Mattson theorem (1969)

Let C be a binary code of length v, minimum weight k.

$$\begin{aligned}
\mathcal{P} &= \{1, 2, \dots, v\}, \\
\mathcal{B} &= \{\mathrm{supp}(x) \mid x \in C,\ \mathrm{wt}(x) = k\}, \\
S &= \{\mathrm{wt}(x) \mid x \in C^{\perp},\ 0 < \mathrm{wt}(x) < v\}, \\
t &= k - |S|.
\end{aligned}$$

Then $(\mathcal{P}, \mathcal{B})$ is a t-(v, k, λ) design for some λ.

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Then $(\mathcal{P}, \mathcal{B})$ is a t-(v, k, λ) design for some λ.

In fact

$$\lambda = \frac{k(k-1)\cdots(k-t+1)}{v(v-1)\cdots(v-t+1)}|\mathcal{B}|.$$

The real vector space of dimension 2^v

From a t-(v,k,λ) design $(\mathcal{P},\mathcal{B})$,

- $p \in \mathcal{P} \longrightarrow e_p$: a unit vector in $\mathbb{F}_2^v$.
- $B \in \mathcal{B} \longrightarrow x^{(B)} \in \mathbb{F}_2^v$: characteristic vector
- $\mathcal{B} \longrightarrow M(\mathcal{D})$: incidence matrix $\longrightarrow C \subset \mathbb{F}_2^v$: binary code

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$V = \mathbb{R}^{2^v} = \mathbb{R}^{\mathbb{F}_2^v}$.

- $x \in \mathbb{F}_2^v \to \hat{x}$: a unit vector in V
- $B \to x^{(B)} \in \mathbb{F}_2^v \to \widehat{x^{(B)}}$: a unit vector in V
- $\mathcal{B} \to \{x^{(B)} \mid B \in \mathcal{B}\} \to$ characteristic vector in V
- $C \to \hat{C}$: the characteristic vector of C in V

Important $2^v \times 2^v$ matrices

The linear transformation of $V = \mathbb{R}^{2^v}$ which is a key to the argument below is the Hadamard matrix of Sylvester type:

$$H = ((-1)^{x \cdot y})_{x,y \in \mathbb{F}_2^v}.$$

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We use H to investigate the metric space $\mathbb{F}_2^v$ with the Hamming distance

$$d(x, y) = \mathrm{wt}(x - y) = \mathrm{wt}(x + y) \quad (x, y \in \mathbb{F}_2^v).$$

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The i-th distance matrix A_i is defined as

$$A_{\color{red}{i}} = (\delta_{d(x,y),\color{red}{i}})_{x,y \in \mathbb{F}_2^v} \quad (0 \le i \le v).$$

A_i: the i-th distance matrix

$$\begin{aligned} A_0 &= I, \\ A_1 A_i &= (i+1)A_{i+1} + (v-i+1)A_{i-1} \quad (1 \le i < v). \end{aligned}$$

In particular, A_i is a polynomial of degree i in A_1.

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Define the diagonal matrix E_i^* by

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$$= \mathrm{diag}(A_i\hat{0}).$$

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E_i^* is "the projection onto weight-i vectors."

$$E_i^*\mathbf{1} = A_i\hat{0}, \qquad \text{where } \mathbf{1} = (1,1,\ldots,1)^\top \in V.$$

$$E_i^* E_j^* = \delta_{i,j} E_i^*, \quad \sum_{i=0}^{v} E_i^* = I.$$

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Theorem (Assmus–Mattson)

Let C be a binary code of length v,

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Design property expressed by matrices

- $T \subset \mathcal{P}$, $|T| = t$, $x^{(T)} \in \mathbb{F}_2^v$: the characteristic vector of T,
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|\{B \in \mathcal{B} \mid T \subset B\}| &= |\{x \in C_k \mid T \subset \mathrm{supp}(x)\}| \\
&= |\{x \in C_k \mid d(x^{(T)}, x) = k - t\}| \\
&= |\{x \in C \mid d(x^{(T)}, x) = k - t\}| - \delta_{k,2t} \\
&= \sum_{x \in C} (A_{k-t})_{x^{(T)},x} - \delta_{k,2t} \\
&= (A_{k-t}\hat{C})_{x^{(T)}} - \delta_{k,2t} \\
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So we want to show

$$E_t^* A_{k-t}\hat{C} \text{ is a constant multiple of } E_t^* \mathbf{1}.$$

$E_t^* A_{k-t} \hat{C} = \lambda E_t^* \mathbf{1}$

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$$\hat{C} = E_0^* \hat{C} + \sum_{i \geq k} E_i^* \hat{C} \quad \text{(minimum weight} = k),$$

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C and $C^\perp$ are connected by H

$$(H\hat{C})_x = \sum_{y\in C}(-1)^{x\cdot y} = \begin{cases} |C| & \text{if } x \in C^\perp \\ 0 & \text{otherwise} \end{cases} = (|C|\widehat{C^\perp})_x,$$

so

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$$E_i^*\widehat{C^\perp} \neq 0 \iff E_i^*H\hat{C} \neq 0 \iff H^{-1}E_i^*H\hat{C} \neq 0$$
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$S = \{\mathrm{wt}(x) \mid x \in C^\perp,\ 0 < \mathrm{wt}(x) < v\}$

$$\begin{aligned} S &= \{i \mid 0 < i < v,\ E_i^* \widehat{C^\perp} \neq 0\} \\ &= \{i \mid 0 < i < v,\ E_i \hat{C} \neq 0\}. \end{aligned}$$

Since $\sum_{i=0}^{v} E_i = I$,

$$\hat{C} = (E_0 + E_v)\hat{C} + \sum_{i \in S} E_i \hat{C}.$$

Theorem (Assmus–Mattson)

$$\hat{C} = (E_0 + E_v)\hat{C} + \sum_{i \in S} E_i \hat{C} = E_0^* \hat{C} + \sum_{i \geq k} E_i^* \hat{C},$$

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$$\text{and } t = k - |S| \implies E_t^* A_{k-t}\left((E_0 + E_v)\hat{C} + \sum_{i \in S} E_i \hat{C}\right) \in \mathbb{R} E_t^* \mathbf{1}.$$

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H diagonalizes A_1

For $y \in \mathbb{F}_2^v$ with $\mathrm{wt}(y) = i$,

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E_i's are projections onto eigenspaces of A_1

$$\begin{aligned} A_1 E_i &= A_1 (\frac{1}{2^v} H E_i^* H) = \frac{1}{2^v} (A_1 H) E_i^* H \\ &= \frac{1}{2^v} (H \sum_{j=1}^{v} (v - 2j) E_j^*) E_i^* H = \frac{1}{2^v} (v - 2i) H E_i^* H \\ &= (v - 2i) E_i. \end{aligned}$$

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$E_v V = \mathbb{R} \sum_{i=0}^{v} (-1)^i E_i^* \mathbf{1}, \qquad E_0 V = \mathbb{R}\mathbf{1}$ $A_1 E_i = (v - 2i) E_i$, so $A_1 E_i V \subset E_i V$

A_1^j also leave $E_i V$ invariant. Thus

$$\begin{aligned} E_t^* A_1^j (E_0 + E_v) \hat{C} &\in E_t^* A_1^j E_0 V + E_t^* A_1^j E_v V \\ &\subset E_t^* E_0 V + E_t^* E_v V \\ &= \mathbb{R} E_t^* \mathbf{1} + \mathbb{R} E^*{}_t \sum_{i=0}^{v} (-1)^i E_i^* \mathbf{1} \\ &= \mathbb{R} E_t^* \mathbf{1}. \end{aligned}$$

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\end{aligned}$$

Being a polynomial in A_1, the matrices A_{k-t} also has the same property

$$E_t^* A_{k-t}(E_0 + E_v)\hat{C} \in \mathbb{R}E_t^* \mathbf{1}.$$

$E_t^* A_1^j (E_0 + E_v)\hat{C},\ E_t^* A_{k-t}(E_0 + E_v)\hat{C} \in \mathbb{R} E_t^* \mathbf{1}$

Theorem (Assmus–Mattson)

$$(E_0 + E_v)\hat{C} + \sum_{i \in S} E_i \hat{C} = E_0^* \hat{C} + \sum_{i \geq k} E_i^* \hat{C} \text{ and } t = k - |S|$$

$$\Longrightarrow\ E_t^* A_{k-t}(E_0 + E_v)\hat{C} + E_t^* A_{k-t} \sum_{i \in S} E_i \hat{C} \in \mathbb{R} E_t^* \mathbf{1}.$$

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$$E_t^* A_1^j \times (E_0 + E_v)\hat{C} + \sum_{i \in S} E_i \hat{C} = E_0^* \hat{C} + \sum_{i \ge k} E_i^* \hat{C}$$

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$\langle I, A_1, A_1^2, A_1^3, \dots \rangle = \langle I, A_1, A_2, A_3, \dots \rangle$

Also,

$$\begin{aligned} E_t^* A_j E_0^* \hat{C} &= E_t^* A_j \hat{0} \\ &= E_t^* E_j^* \mathbf{1} \\ &= \delta_{t,j} E_t^* \mathbf{1} \\ &\in \mathbb{R} E_t^* \mathbf{1}. \end{aligned}$$

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$V = \bigoplus_{i=0}^{v} E_i V$: eigenspace decomposition of A_1

A_1 has $|S|$ eigenvalues on

$$W = \bigoplus_{i \in S} E_i V.$$

Being a polynomial in A_1, the matrix A_{k-t} has at most $|S|$ eigenvalues on W, so $\exists a_0, \ldots, a_{|S|-1} \in \mathbb{Q}$ such that

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So

$$A_{k-t} \sum_{i \in S} E_i \hat{C} = \sum_{j=0}^{|S|-1} a_j A_1^j \sum_{i \in S} E_i \hat{C}.$$

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Proof:

$$E_t^* A_{k-t} \sum_{i\in S} E_i\hat{C} = E_t^* \sum_{j=0}^{|S|-1} a_j A_1^j \sum_{i\in S} E_i\hat{C} = \sum_{j=0}^{|S|-1} a_j E_t^* A_1^j \sum_{i\in S} E_i\hat{C}$$

$$\equiv \sum_{j=0}^{|S|-1} a_j E_t^* A_1^j \sum_{i\geq k} E_i^*\hat{C} = \sum_{j=0}^{|S|-1} \sum_{i\geq k} a_j (E_t^* A_1^j E_i^*)\hat{C}.$$

$A_{k-t}\sum_{i\in S}E_i\hat{C}=\sum_{j=0}^{|S|-1}a_jA_1^j\sum_{i\in S}E_i\hat{C}$

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and $t=k-|S| \implies E_t^*A_{k-t}\sum_{i\in S}E_i\hat{C}\in\mathbb{R}E_t^*\mathbf{1}.$

Proof:

$$E_t^*A_{k-t}\sum_{i\in S}E_i\hat{C}=E_t^*\sum_{j=0}^{|S|-1}a_jA_1^j\sum_{i\in S}E_i\hat{C}=\sum_{j=0}^{|S|-1}a_jE_t^*A_1^j\sum_{i\in S}E_i\hat{C}$$

$$\equiv\sum_{j=0}^{|S|-1}a_jE_t^*A_1^j\sum_{i\geq k}E_i^*\hat{C}=\sum_{j=0}^{|S|-1}\sum_{i\geq k}a_j(E_t^*A_1^jE_i^*)\hat{C}.$$

$A_{k-t} \sum_{i \in S} E_i \hat{C} = \sum_{j=0}^{|S|-1} a_j A_1^j \sum_{i \in S} E_i \hat{C}$

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$$E_t^* A_{k-t} \sum_{i \in S} E_i \hat{C} = E_t^* \sum_{j=0}^{|S|-1} a_j A_1^j \sum_{i \in S} E_i \hat{C} = \sum_{j=0}^{|S|-1} a_j E_t^* A_1^j \sum_{i \in S} E_i \hat{C}$$

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Theorem (Assmus–Mattson)

$$E_t^* A_1^j \sum_{i\in S} E_i \hat{C} \equiv E_t^* A_1^j \sum_{i\geq k} E_i^* \hat{C} \pmod{\mathbb{R} E_t^* \mathbf{1}} \quad (\forall j)$$

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The Assmus–Mattson theorem

Theorem

Let C be a binary code of length v, minimum weight k.

$$\begin{aligned}
\mathcal{P} &= \{1, 2, \ldots, v\}, \\
\mathcal{B} &= \{\mathrm{supp}(x) \mid x \in C,\ \mathrm{wt}(x) = k\}, \\
S &= \{\mathrm{wt}(x) \mid x \in C^{\perp},\ 0 < \mathrm{wt}(x) < v\}, \\
t &= k - |S|.
\end{aligned}$$

Then $(\mathcal{P}, \mathcal{B})$ is a t-(v, k, λ) design for some λ.

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- C: $[24, 12, 8]$ binary doubly even self-dual ($C = C^{\perp}$) code, so $k = 8$ and C has only weights $0, 8, 12, 16, 24$.

$$\begin{aligned}
S &= \{\mathrm{wt}(x) \mid x \in C^{\perp},\ 0 < \mathrm{wt}(x) < 24\} = \{8, 12, 16\},\\
t &= k - |S| = 8 - 3 = 5.
\end{aligned}$$

Uniqueness of the extended binary Golay code

C: $[24, 12, 8]$ binary doubly even self-dual ($C = C^\perp$) code.

- The Assmus–Mattson theorem implies $(\mathcal{P}, \mathcal{B})$ is a 5-$(24, 8, \lambda)$ design, where $\mathcal{P} = \{1, 2, \ldots, 24\}$,

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- So C is the binary code of a 5-$(24, 8, 1)$ design which was already shown to be unqiue.

This proves the uniqueness of the extended binary Golay code.

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suppose $C = C^\perp$, doubly even $\Longrightarrow$ S not too large

Binary doubly even self-dual codes

Under what circumstance can one obtain a 5-design from a doubly even self-dual code? Let k be the minimum weight.

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In general, $\forall k$: a multiple of 4, $|S| = k - 5$,

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$v = 6k - 24 = 24m$, where $k = 4m + 4$.

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For $m \geq 1$, an extremal binary doubly even self-dual code gives a 5-$(24m, 4m + 4, \lambda)$ design by the Assmus–Mattson theorem.

- $m = 1$: the extended binary Golay code and the 5-$(24, 8, 1)$ design
- $m = 2$: Houghten–Lam–Thiel–Parker (2003): unique $[48, 24, 12]$ code and a 5-$(48, 12, 8)$ design which is unique under self-orthogonality.

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- For $m \geq 1$, an extremal binary doubly even self-dual code gives a 5-$(24m, 4m + 4, \lambda)$ design by the Assmus–Mattson theorem.
- For $m \geq 3$, neither a code nor a design is known.

Theorem (Zhang, 1999)

There does not exist an extremal $[24m, 12m, 4m + 4]$ binary doubly even self-dual code for $m \geq 154$.