Codes Generated by Designs, and Designs Supported by Codes
Part III

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Summary of Part I and II

$\mathcal{D}$: 5-$(24, 8, 1)$ design (Witt system).

- The binary code C of $\mathcal{D}$ is a doubly even self-dual $[24, 12, 8]$ code.
- $\{\mathrm{supp}(x) \mid x \in C,\ \mathrm{wt}(x) = 8\} = \mathcal{B}$.
- There is a unique 5-$(24, 8, 1)$ design up to isomorphism.
- There is a unique doubly even self-dual $[24, 12, 8]$ code (up to isomorphism), by the Assmus–Mattson theorem.

Part III will cover

- Hadamard matrices
- Characterization of (binary) Hadamard matrices "contained" in the doubly even self-dual $[24, 12, 8]$ code, and their relationships to ternary self-dual codes

Hadamard matrices

Definition

A Hadamard matrix of order n is an $n \times n$ matrix with entries ± 1, such that rows are pairwise orthogonal:

- H: $n \times n$ matrix,
- $H_{i,j} \in \{\pm 1\}$ for all $i, j \in \{1, \dots, n\}$,
- $HH^\top = nI$.

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Example

The Hadamard matrix of Sylvester type, where $n = 2^v$:

$$H = ((-1)^{x \cdot y})_{x,y \in \mathbb{F}_2^v}.$$

$$v = 1 \implies H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Hadamard matrices of Sylvester type, $n = 2^v$

$$v = 2 \implies H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Hadamard matrices of Sylvester type, $n = 2^v$

$$v = 2 \implies H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$v = 3 \implies H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \end{bmatrix}$$

Existence of Hadamard matrices

A Hadamard matrix of order n exists for

$$n = 1, 2, 4, 8, 12, 16, \dots \text{(multiples of 4)}, \dots, 664, \dots$$

Except $n = 1, 2$, the existence of a Hadamard matrix of order n implies $n \equiv 0 \pmod 4$:

$$\begin{array}{cccc}
1 \cdots 1 & 1 \cdots 1 & 1 \cdots 1 & 1 \cdots 1 \\
1 \cdots 1 & 1 \cdots 1 & -1 \cdots -1 & -1 \cdots -1 \\
1 \cdots 1 & -1 \cdots -1 & 1 \cdots 1 & -1 \cdots -1
\end{array}$$

But it is not known whether a Hadamard matrix of order 668 exists.

Conjecture

A Hadamard matrix of order n exists for any $n \equiv 0 \pmod 4$.

Sylvester type: $n = 2^v = 1, 2, 4, 8, 16, \dots$.

Classification of Hadamard matrices

If H is a Hadamard matrix, then so is $H^{\top}$.
Two Hadamard matrices are said to be equivalent if one is obtained from the other by row or column permutations or negations:

$$H_1 \cong H_2 \iff \exists P, Q,\ PH_1Q = H_2,$$

where P and Q are matrices in which only 1 or -1 appear exactly once in every row and once in every column, all other entries are 0.

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The numbers of equivalence classes of Hadamard matrices are known for orders up to 32.

order	1	2	4	8	12	16	20	24	28	32
number	1	1	1	1	1	5	3	60	487	13,710,027

16, 20: Hall; 24: Ito–Leon–Longyear, Kimura; 28: Kimura, Spence; 32: Kharaghani and Tayfeh-Rezaie (2012).

Normalized and binary Hadamard matrices

Every Hadamard matrix is equivalent to the one with 1 everywhere in the first row:

$$H = \begin{bmatrix} 1 & 1 \cdots 1 \\ & \cdots \\ & \pm 1 \\ & \cdots \end{bmatrix}$$

Such a Hadamard matrix H is said to be normalized.

Normalized and binary Hadamard matrices

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$$B = \frac{1}{2}(H + J) = \begin{bmatrix} 1 & 1 \cdots 1 \\ & \cdots \\ & 1 \text{ or } 0 \\ & \cdots \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ b^{(1)} \\ \vdots \\ b^{(n-1)} \end{bmatrix}$$

where $J =$ all one matrix.

Hadamard 3-design

- H: a normalized Hadamard matrix of order n.
- $B = \frac{1}{2}(H + J)$: the associated binary Hadamard matrix. B has row vectors $b^{(0)} = \mathbf{1}$, $b^{(1)}, \dots, b^{(n-1)}$.

$$\mathcal{P} = \{1, \dots, n\},$$
$$\mathcal{B} = \bigcup_{i=1}^{n-1} \{\mathrm{supp}(b^{(i)}), \mathrm{supp}(\mathbf{1} + b^{(i)})\}.$$

Then $(\mathcal{P}, \mathcal{B})$ is a 3-$(n, \frac{n}{2}, \frac{n}{4} - 1)$ design. Indeed, consider the transpose of

$$\begin{matrix} 1 & 1 \cdots 1 & 1 \cdots 1 & 1 \cdots 1 & 1 \cdots 1 \\ 1 & 1 \cdots 1 & 1 \cdots 1 & -1 \cdots -1 & -1 \cdots -1 \\ 1 & \underbrace{1 \cdots 1}_{\frac{4}{n}-1} & \underbrace{-1 \cdots -1}_{\frac{4}{n}} & \underbrace{1 \cdots 1}_{\frac{4}{n}} & \underbrace{-1 \cdots -1}_{\frac{4}{n}} \end{matrix} \quad \text{in} \quad \begin{bmatrix} H \\ -H \end{bmatrix}$$

The isomorphism class of Hadamard 3-design

Definition

Two designs $(\mathcal{P},\mathcal{B})$ and $(\mathcal{P}',\mathcal{B}')$ are said to be isomorphic if there is a bijection from $\mathcal{P}$ to $\mathcal{P}'$ which maps $\mathcal{B}$ to $\mathcal{B}'$.

$$\begin{tikzcd}
H \arrow[r, "\text{normalize}"] \arrow[d, "\text{swap rows}"'] & B \arrow[r] & (\mathcal{P},\mathcal{B}) \\
H' \arrow[r, "\text{normalize}"] & B' \arrow[r] & (\mathcal{P},\mathcal{B}')
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Definition

The binary code of a Hadamard matrix H is defined as that of the Hadamard 3-design $(\mathcal{P},\mathcal{B})$ obtained from the binary Hadamard matrix associated to any normalized Hadamard matrix equivalent to H.

Is it well defined?

The isomorphism class of the binary code

$$H = \begin{bmatrix} \mathbf{1} \\ h^{(1)} \\ \vdots \\ h^{(n-1)} \end{bmatrix} \longrightarrow B = \frac{1}{2}(H + J) = \begin{bmatrix} \mathbf{1} \\ b^{(1)} \\ \vdots \\ b^{(n-1)} \end{bmatrix}$$

$$\begin{bmatrix} h^{(1)} \\ \mathbf{1} \\ \vdots \\ h^{(n-1)} \end{bmatrix} \to H' = \begin{bmatrix} \mathbf{1} \\ h^{(1)} \\ h^{(2)} * h^{(1)} \\ \vdots \\ h^{(n-1)} * h^{(1)} \end{bmatrix} \to B' = \frac{1}{2}(H' + J) = \begin{bmatrix} \mathbf{1} \\ b^{(1)} \\ b^{(2)} + b^{(1)} + \mathbf{1} \\ \vdots \\ b^{(n-1)} + b^{(1)} + \mathbf{1} \end{bmatrix}$$

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1	1	1	1	1	1
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B and B' generate the same binary code

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Normalized and binary Hadamard matrices

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Then

- The binary code of H is generated by B.
- B has first row $\mathbf{1}$, the vector with weight n.
- All the other rows have weight $\frac{n}{2}$.
- Two distinct rows of weight $\frac{n}{2}$ have $\frac{n}{4}$ coordinates in common in their supports.
- $n \equiv 0 \pmod 8 \implies$ the binary code of H is self-orthogonal.

The binary code of a Hadamard matrix

Lemma

Let C be the binary code of a Hadamard matrix of order n.

- If $n \equiv 0 \pmod 8$, then C is doubly even self-orthogonal.
- If $n \equiv 8 \pmod{16}$, then C is doubly even self-dual.

In particular, for $n = 24$, C is doubly even self-dual.

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In particular, for $n = 24$, C is doubly even self-dual.

- One can ask: which of the 60 Hadamard matrices of order 24 give the extended binary Golay code?
- Among the 60 Hadamard matrices of order 24, only two give the extended binary Golay code.

Ternary codes

A (linear) ternary code of length n is a subspace of the vector space $\mathbb{F}_3^n$. If C is a ternary code and $\dim C = k$, we say C is an ternary $[n, k]$ code. The dual code of a ternary code C is defined as

$$C^\perp = \{x \in \mathbb{F}_3^n \mid x \cdot y = 0 \ (\forall y \in C)\}.$$

where

$$x \cdot y = \sum_{i=1}^{n} x_i y_i.$$

Then $\dim C^\perp = n - \dim C$. The code C is said to be self-orthogonal if $C \subset C^\perp$ and self-dual if $C = C^\perp$.

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Generator matrix of a ternary code

If a ternary code C of length n is generated by row vectors $x^{(1)}, \dots, x^{(m)}$, then the matrix

$$\begin{bmatrix} x^{(1)} \\ \vdots \\ x^{(m)} \end{bmatrix}$$

is called a generator matrix of C. This means

$$C = \{\sum_{i=1}^{m} \epsilon_i x^{(i)} \mid \epsilon_1, \dots, \epsilon_m \in \mathbb{F}_3\} \subset \mathbb{F}_3^n.$$

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Definition

The ternary code of a Hadamard matrix H is the ternary code with generator matrix H.

Weight

For $x \in \mathbb{F}_3^n$, we write

$$\begin{aligned} \operatorname{supp}(x) &= \{i \mid 1 \le i \le n,\ x_i \ne 0\}, \\ \operatorname{wt}(x) &= |\operatorname{supp}(x)|. \end{aligned}$$

For a ternary code C, its minimum weight is

$$\min\{\operatorname{wt}(x) \mid 0 \ne x \in C\}.$$

If an $[n, k]$ ternary code C has minimum weight d, we call C an $[n, k, d]$ code.

Ternary self-dual codes of length 24

Lemma

Let n be an integer divisible by 4. If $3|n$ and $9 \nmid n$, then the ternary code of a Hadamard matrix of order n is self-dual.

In particular, the ternary code of a Hadamard matrix of order 24 is self-dual.

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In particular, the ternary code of a Hadamard matrix of order 24 is self-dual.

- Leon–Pless–Sloane (1981): there are two self-dual codes of length 24 with minimum weight 9 (largest possible), up to isomorphism.
- One can ask: which of the 60 Hadamard matrices of order 24 give the codes with minimum weight 9?
- Among the 60 Hadamard matrices of order 24, only two give codes with minimum weight 9.

Verification using MAGMA

Assmus and Key in their 1992 book observed:

```
DB:=HadamardDatabase();
NumberOfMatrices(DB,24) eq 60;
H24s:=[Matrix(DB,24,i):i in [1..60]];
normalize:=func<H|H*DiagonalMatrix(Eltseq(H[1]))>;
J:=Matrix(Integers(),24,24,[1:i in [1..24^2]]);
bH:=func<H|Parent(H)![x div 2:x in Eltseq(normalize(H)+J)]>;
bC:=func<H|LinearCode(ChangeRing(bH(H),GF(2)))>;
tCT:=func<H|LinearCode(ChangeRing(Transpose(H),GF(3)))>;
[i:i in [1..60]|MinimumWeight(bC(H24s[i])) eq 8] eq [3,9];
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[i:i in [1..60]|MinimumWeight(bC(H24s[i])) eq 8] eq [3,9];
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Verification using MAGMA

Assmus and Key in their 1992 book observed:

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DB:=HadamardDatabase();
NumberOfMatrices(DB,24) eq 60;
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normalize:=func<H|H*DiagonalMatrix(Eltseq(H[1]))>;
J:=Matrix(Integers(),24,24,[1:i in [1..24^2]]);
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Assmus and Key, 1992

Fact

Let H be a Hadamard matrix of order 24. The following are equivalent.

- The binary code of H has minimum weight 8 (largest).
- The ternary code of $H^\top$ has minimum weight 9 (largest).

- The binary code of H is doubly even self-dual, and the minimum weight is 4 or 8.
- The ternary code of $H^\top$ is self-dual, and the minimum weight is 6 or 9. (A ternary self-dual code may have minimum weight 3, but no ternary code of a Hadamard matrix has minimum weight 3).
- There are two (up to equivalence) Hadamard matrices H satisfying the above equivalent conditions.

H: a normalized Hadamard matrix of order 24

- C_2: the binary code of $H =$ the binary code with generator matrix $B = \frac{1}{2}(H + J)$.
- C_3: the ternary code of $H^\top$.
- C_2 is doubly even self-dual, and C_3 is self-dual.
- C_2 has only weights divisible by 4, C_3 has only weights divisible by 3.

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The following are equivalent:

- C_2 has minimum weight 8 (largest).
- C_3 has minimum weight 9 (largest).

We first show: $C_3 = C_3^\perp$ has no vectors of weight 3.

C_3: the ternary code of $H^\top$ (n is arbitrary)

Suppose

$$\begin{aligned} C_3^\perp &= \{v \in \mathbb{F}_3^n \mid H^\top v^\top = 0\} \\ &= \{v \bmod 3 \mid v \in \mathbb{Z}^n,\ vH \equiv 0 \pmod 3\} \end{aligned}$$

contains a vector v of weight 3:

$$v = (0, \dots, 0, \epsilon_i, 0, \dots, 0, \epsilon_j, 0, \dots, 0, \epsilon_k, 0, \dots, 0)$$

where $\epsilon_i, \epsilon_j, \epsilon_k \in \{\pm 1\}$.

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$$\begin{aligned} & vH \equiv 0 \pmod 3 \\ \implies & \epsilon_i H_{i,\ell} + \epsilon_j H_{j,\ell} + \epsilon_k H_{k,\ell} \equiv 0 \pmod 3 \quad (\forall \ell \in \{1, \dots, n\}) \\ \implies & \epsilon_i H_{i,\ell} = \epsilon_j H_{j,\ell} = \epsilon_k H_{k,\ell} \quad (\forall \ell \in \{1, \dots, n\}) \\ \implies & \epsilon_j \epsilon_i H_{i,\ell} = H_{j,\ell} \quad (\forall \ell \in \{1, \dots, n\}) \\ \implies & \text{row } i \text{ of } H = \text{row } j \text{ of } H\text{, up to sign} \end{aligned}$$

This is impossible for a Hadamard matrix H.

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$C_3^\perp$ does not have weight 3

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Fact

The following are equivalent:

- C_2 has minimum weight 8 (i.e., C_2 doesn't have weight 4)
- C_3 has minimum weight 9 (i.e., C_3 doesn't have weight 6)

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The following are equivalent.

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Proof of ($2 \implies 1$). $v \in \{0, \pm 1\}^{24} \subset \mathbb{Z}^{24}$, $\mathrm{wt}(v) = 6$, $vH \equiv 0 \pmod 3$. Set

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Hadamard matrices and norms

Lemma

Let H be a Hadamard matrix of order n, v a vector in $\mathbb{Z}^n$. Then

- $vH \equiv \|v\|^2\mathbf{1} \pmod 2$,
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Proof.

$$vH \equiv vJ = (\sum_{i=1}^{n} v_i)\mathbf{1} \equiv (\sum_{i=1}^{n} v_i^2)\mathbf{1} = \|v\|^2\mathbf{1} \pmod 2.$$

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$$\|vH\|^2 = vHH^\top v^\top = v(nI)v^\top = n\|v\|^2.$$

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$$u = \frac{1}{6}vH \implies u \in \mathbb{Z}^{24},\ \mathrm{wt}(u \bmod 2) = 4,\ u \bmod 2 \in C_2.$$

Lemma

- $vH \equiv \|v\|^2\mathbf{1} \pmod 2$,
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Since $\|v\|^2 = \mathrm{wt}(v) = 6$, $vH \equiv \|v\|^2\mathbf{1} \equiv 0 \pmod 2$. Thus $vH \equiv 0 \pmod 6$, and $u \in \mathbb{Z}^{24}$.

$$\|u\|^2 = \frac{1}{6^2}24\|v\|^2 = \frac{24 \cdot 6}{6^2} = 4 \implies \mathrm{wt}(u \bmod 2) = 4.$$

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H: a normalized Hadamard matrix of order 24

- $C_2 = \{wB \bmod 2 \mid w \in \mathbb{Z}^{24}\}$: the binary code of H, $B = \frac{1}{2}(H + J)$.
- $v \in \{0, \pm 1\}^{24} \subset \mathbb{Z}^{24}$, $\mathrm{wt}(v) = 6$, $vH \equiv 0 \pmod 3$.

$$u = \frac{1}{6}vH \implies u \in \mathbb{Z}^{24},\ \mathrm{wt}(u \bmod 2) = 4,\ u \bmod 2 \in C_2.$$

$$\begin{aligned} u &\equiv 3u \pmod 2 \\ &= \frac{3}{6}vH \\ &= \frac{1}{2}v(2B - J) = vB - \frac{1}{2}vJ \\ &\equiv vB + \epsilon\mathbf{1} \pmod 2 \quad (\epsilon \in \{0, 1\}) \\ &= (v + \epsilon e_1)B \in C_2 \quad \text{(after reducing mod 2).} \end{aligned}$$

H: a normalized Hadamard matrix of order 24

- C_2: the binary code of $H =$ the binary code with generator matrix $B = \frac{1}{2}(H + J)$.
- C_3: the ternary code of $H^\top$.

Then C_2 is doubly even self-dual, and C_3 is self-dual.

Theorem (Munemasa–Tamura, 2012)

The following are equivalent:

1. C_2 has minimum weight 8 (largest).
2. C_3 has minimum weight 9 (largest).

We have proved $1 \implies 2$ by showing its contrapositive assertion. The other implication can be proved similarly.

$u = \frac{1}{6}vH \iff v = \frac{1}{4}uH^\top$

H: a normalized Hadamard matrix of order 48

Similarly, one can consider a code over $\mathbb{Z}/4\mathbb{Z}$, the ring of integers modulo 4. The Euclidean weight of a vector $v \in (\mathbb{Z}/4\mathbb{Z})^n$ is

$$\mathrm{wt}(v) = \sum_{i=1}^{n} v_i^2,$$

where we regard $v_i \in \{0, \pm 1, 2\} \subset \mathbb{Z}$.

Theorem

- C_4: the code over $\mathbb{Z}/4\mathbb{Z}$ with generator matrix $B = \frac{1}{2}(H + J)$.
- C_3: the ternary code of $H^\top$.

Then both C_4 and C_3 are self-dual. Moreover, the following are equivalent:

- C_4 has minimum Euclidean weight 24 (largest).
- C_3 has minimum weight 15 (largest).

Hadamard matrices of order 48 and ternary codes

Theorem

If C is a ternary self-dual code of length 48 and minimum weight 15, then C is generated by a Hadamard matrix.

Unlike the case $n = 24$, the following problem is still open.

Problem

Classify ternary self-dual codes of length 48 with minimum weight 15, or classify Hadamard matrices of order 48 which generate such a code.

Extremal ternary self-dual codes

Theorem (Mallows–Sloane, 1973)

For $m \geq 1$, a ternary self-dual $[12m, 6m]$ code has minimum weight at most $3m + 3$.

Definition

A ternary self-dual $[12m, 6m]$ code with minimum weight $3m + 3$ is called extremal.

- $m = 1$: the extended ternary Golay code and the 5-$(12, 6, 1)$ design,
- $m = 2$: exactly two codes,
- $m = 3$: at least one code,
- $m = 4$: at least two codes,
- $m = 5$: at least two codes.

All these codes are generated by a Hadamard matrix.

Extremal ternary self-dual codes

Definition

A ternary self-dual $[12m, 6m]$ code with minimum weight $3m + 3$ is called extremal.

For $m \geq 6$, no code is known. In fact, for m even and $m \geq 6$, an extremal ternary self-dual $[12m, 6m, 3m + 3]$ code does not exist.

Theorem (Shengyuan Zhang, 1999)

There does not exist an extremal $[12m, 6m, 3m + 3]$ ternary self-dual code for $m \geq 70$.