Complex Hadamard matrices and 3-class association schemes

Akihiro Munemasa[1]

[1]Graduate School of Information Sciences
Tohoku University

(joint work with Takuya Ikuta)
June 26, 2013
The 30th Algebraic Combinatorics Symposium
Shizuoka University

Hadamard matrices and generalizations

- A (real) Hadamard matrix of order n is an $n \times n$ matrix H with entries ± 1, satisfying $HH^\top = nI$.

Conjecture

$\exists$ Hadamard matrix of order n whenever $n \equiv 0 \pmod 4$.

Hadamard matrices and generalizations

- A (real) Hadamard matrix of order n is an $n \times n$ matrix H with entries ± 1, satisfying $HH^\top = nI$.

Conjecture

$\exists$ Hadamard matrix of order n whenever $n \equiv 0 \pmod 4$.

- A complex Hadamard matrix of order n is an $n \times n$ matrix H with entries in $\{\zeta \in \mathbb{C} \mid |\zeta| = 1\}$, satisfying $HH^* = nI$, where $*$ means the conjugate transpose.

Hadamard matrices and generalizations

- A (real) Hadamard matrix of order n is an $n \times n$ matrix H with entries ± 1, satisfying $HH^\top = nI$.

Conjecture

$\exists$ Hadamard matrix of order n whenever $n \equiv 0 \pmod 4$.

- A complex Hadamard matrix of order n is an $n \times n$ matrix H with entries in $\{\zeta \in \mathbb{C} \mid |\zeta| = 1\}$, satisfying $HH^* = nI$, where $*$ means the conjugate transpose.

We propose a strategy to construct infinite families of complex Hadamard matrices using association schemes, and demonstrate a successful case.

Circulant (complex) Hadamard matrices

$$H = \begin{bmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \\ & \ddots & \ddots & \ddots \\ a_1 & & & a_0 \end{bmatrix} = \sum_{i=0}^{n-1} a_i A^i, \quad A = \begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & \ddots \\ 1 & & & \end{bmatrix}$$

Instead of n^2 entries, there are only n entries to determine.

Circulant (complex) Hadamard matrices

$$H = \begin{bmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \\ & \ddots & \ddots & \ddots \\ a_1 & & & a_0 \end{bmatrix} = \sum_{i=0}^{n-1} a_i A^i, \quad A = \begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & \ddots \\ 1 & & & \end{bmatrix}$$

Instead of n^2 entries, there are only n entries to determine.

Björck–Fröberg (1991–1992) circulant Hadamard, $n \leq 8$

Faugère (2001), (2004) circulant Hadamard, $n = 9, 10$

Circulant (complex) Hadamard matrices

$$H = \begin{bmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \\ & \ddots & \ddots & \ddots \\ a_1 & & & a_0 \end{bmatrix} = \sum_{i=0}^{n-1} a_i A^i, \quad A = \begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & \ddots \\ 1 & & & \end{bmatrix}$$

Instead of n^2 entries, there are only n entries to determine.

Björck–Fröberg (1991–1992) circulant Hadamard, $n \le 8$

Faugère (2001), (2004) circulant Hadamard, $n = 9, 10$

On the other hand, it is conjectured that no circulant real Hadamard matrix of order > 4 exists.

Goethals–Seidel (1970) symmetric regular (real) Hadamard matrix necessarily comes from a strongly regular graph (SRG) on $4s^2$ vertices

de la Harpe–Jones (1990) SRG n: prime $\equiv 1 \pmod 4 \rightarrow$ symmetric circulant complex Hadamard

Goethals–Seidel (1970) symmetric regular (real) Hadamard matrix necessarily comes from a strongly regular graph (SRG) on $4s^2$ vertices

de la Harpe–Jones (1990) SRG n: prime $\equiv 1 \pmod 4 \rightarrow$ symmetric circulant complex Hadamard

M.–Watatani (1992) DRT n: prime $\equiv 3 \pmod 4 \rightarrow$ non-symmetric circulant complex Hadamard

Goethals–Seidel (1970) symmetric regular (real) Hadamard matrix necessarily comes from a strongly regular graph (SRG) on $4s^2$ vertices

de la Harpe–Jones (1990) SRG n: prime $\equiv 1 \pmod 4 \rightarrow$ symmetric circulant complex Hadamard

M.–Watatani (1992) DRT n: prime $\equiv 3 \pmod 4 \rightarrow$ non-symmetric circulant complex Hadamard

$$H = \alpha_0 A_0 + \alpha_1 A_1 + \alpha_2 A_2, \quad A_0 = I$$

Goethals–Seidel (1970) symmetric regular (real) Hadamard matrix necessarily comes from a strongly regular graph (SRG) on $4s^2$ vertices

de la Harpe–Jones (1990) SRG n: prime $\equiv 1 \pmod 4 \to$ symmetric circulant complex Hadamard

M.–Watatani (1992) DRT n: prime $\equiv 3 \pmod 4 \to$ non-symmetric circulant complex Hadamard

$$H = \alpha_0 A_0 + \alpha_1 A_1 + \alpha_2 A_2, \quad A_0 = I$$

$$A_1^\top = A_1,\ A_2^\top = A_2 \quad \text{(G.–J., de la H.–J.)}$$

$$A_1^\top = A_2, \quad \text{(M.–W.)}$$

Goethals–Seidel (1970) symmetric regular (real) Hadamard matrix necessarily comes from a strongly regular graph (SRG) on $4s^2$ vertices

de la Harpe–Jones (1990) SRG n: prime $\equiv 1 \pmod 4 \to$ symmetric circulant complex Hadamard

M.–Watatani (1992) DRT n: prime $\equiv 3 \pmod 4 \to$ non-symmetric circulant complex Hadamard

$$H = \alpha_0 A_0 + \alpha_1 A_1 + \alpha_2 A_2, \quad A_0 = I$$

$$A_1^\top = A_1,\ A_2^\top = A_2 \quad \text{(G.–J., de la H.–J.)}$$

$$A_1^\top = A_2, \quad \text{(M.–W.)}$$

Unifying principle: association schemes.
(strongly regular graphs is a special case)

Godsil–Chan (2010), and Chan (2011) classified complex Hadamard matrices of the form:

$$H = \alpha_0 I + \alpha_1 A_1 + \alpha_2 A_2 \quad \text{(we may assume } \alpha_0 = 1\text{)}$$

where $|\alpha_1| = |\alpha_2| = 1$, and

$$A_1 = \text{adjacency matrix of a SRG } \Gamma,$$
$$A_2 = \text{adjacency matrix of } \overline{\Gamma}.$$

Godsil–Chan (2010), and Chan (2011) classified complex Hadamard matrices of the form:

$$H = \alpha_0 I + \alpha_1 A_1 + \alpha_2 A_2 \quad \text{(we may assume } \alpha_0 = 1)$$

where $|\alpha_1| = |\alpha_2| = 1$, and

$$A_1 = \text{adjacency matrix of a SRG } \Gamma,$$
$$A_2 = \text{adjacency matrix of } \overline{\Gamma}.$$

Godsil–Chan (2010), and Chan (2011) classified complex Hadamard matrices of the form:

$$H = \alpha_0 I + \alpha_1 A_1 + \alpha_2 A_2 \quad \text{(we may assume } \alpha_0 = 1\text{)}$$

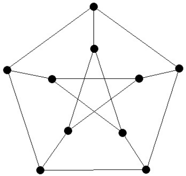
where $|\alpha_1| = |\alpha_2| = 1$, and

$$A_1 = \text{adjacency matrix of a SRG } \Gamma,$$
$$A_2 = \text{adjacency matrix of } \overline{\Gamma}.$$

also found a complex Hadamard matrix of the form

$$H = I + \alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3$$

of order 15 from the line graph $L(O_3)$ of the Petersen graph O_3.



The Bose–Mesner algebra of a symmetric association scheme of class d

$$\langle A_0, A_1, \ldots, A_d \rangle = \langle E_0, E_1, \ldots, E_d \rangle$$

is a commutative semisimple algebra with primitive idempotents $E_0, E_1, \ldots, E_d$.

$$\sum_{i=0}^{d} A_i = J, \quad \sum_{i=0}^{d} E_i = I, \quad A_0 = I,\ nE_0 = J.$$

$$A_j \circ A_k = \delta_{jk} A_j, \quad E_j E_k = \delta_{jk} E_j$$

The Bose–Mesner algebra of a symmetric association scheme of class d

$$\langle A_0, A_1, \ldots, A_d \rangle = \langle E_0, E_1, \ldots, E_d \rangle$$

is a commutative semisimple algebra with primitive idempotents $E_0, E_1, \ldots, E_d$.

$$\sum_{i=0}^{d} A_i = J, \quad \sum_{i=0}^{d} E_i = I, \quad A_0 = I,\ nE_0 = J.$$

$$A_j \circ A_k = \delta_{jk} A_j, \quad E_j E_k = \delta_{jk} E_j$$

$$A_j = \sum_{i=0}^{d} p_{ij} E_i, \quad A_i^\top = A_i \quad (\forall i), \implies p_{ji} \in \mathbb{R}.$$

The Bose–Mesner algebra of a symmetric association scheme of class d

$$\langle A_0, A_1, \dots, A_d \rangle = \langle E_0, E_1, \dots, E_d \rangle$$

is a commutative semisimple algebra with primitive idempotents $E_0, E_1, \dots, E_d$.

$$\sum_{i=0}^{d} A_i = J, \quad \sum_{i=0}^{d} E_i = I, \quad A_0 = I,\ nE_0 = J.$$

$$A_j \circ A_k = \delta_{jk} A_j, \quad E_j E_k = \delta_{jk} E_j$$

$$A_j = \sum_{i=0}^{d} p_{ij} E_i, \quad A_i^\top = A_i \quad (\forall i), \implies p_{ji} \in \mathbb{R}.$$

$$A_i = \sum_{k=0}^{d} p_{ki} E_k, \quad E_j E_k = \delta_{jk} E_j, \quad p_{ki} \in \mathbb{R}.$$

When is $H = \sum_{i=0}^{d} \alpha_i A_i$ with $|\alpha_i| = 1$ a complex Hadamard matrix?

$HH^* = nI \iff$

$\iff$

$\iff$

$\iff$

$$A_i = \sum_{k=0}^{d} p_{ki} E_k, \quad E_j E_k = \delta_{jk} E_j, \quad p_{ki} \in \mathbb{R}.$$

When is $H = \sum_{i=0}^{d} \alpha_i A_i$ with $|\alpha_i| = 1$ a complex Hadamard matrix?

$$H\overline{H} = nI \iff \left(\sum_{i=0}^{d} \alpha_i \sum_{k=0}^{d} p_{ki} E_k\right)\overline{\left(\sum_{j=0}^{d} \alpha_j \sum_{k=0}^{d} p_{kj} E_k\right)} = nI$$

$$\iff$$

$$\iff$$

$$\iff$$

$$A_i = \sum_{k=0}^{d} p_{ki} E_k, \quad E_j E_k = \delta_{jk} E_j, \quad p_{ki} \in \mathbb{R}.$$

When is $H = \sum_{i=0}^{d} \alpha_i A_i$ with $|\alpha_i| = 1$ a complex Hadamard matrix?

$$H\overline{H} = nI \iff (\sum_{i=0}^{d} \alpha_i \sum_{k=0}^{d} p_{ki} E_k)\overline{(\sum_{j=0}^{d} \alpha_j \sum_{k=0}^{d} p_{kj} E_k)} = nI$$

$$\iff$$

$$\iff$$

$$\iff$$

$$A_i = \sum_{k=0}^{d} p_{ki} E_k, \quad E_j E_k = \delta_{jk} E_j, \quad p_{ki} \in \mathbb{R}.$$

When is $H = \sum_{i=0}^{d} \alpha_i A_i$ with $|\alpha_i| = 1$ a complex Hadamard matrix?

$$H = nI \iff \left(\sum_{i=0}^{d} \alpha_i \sum_{k=0}^{d} p_{ki} E_k\right)\left(\sum_{j=0}^{d} \overline{\alpha_j} \sum_{k=0}^{d} p_{kj} E_k\right) = nI$$

$$\iff$$

$$\iff$$

$$\iff$$

$$A_i = \sum_{k=0}^{d} p_{ki} E_k, \quad E_j E_k = \delta_{jk} E_j, \quad p_{ki} \in \mathbb{R}.$$

When is $H = \sum_{i=0}^{d} \alpha_i A_i$ with $|\alpha_i| = 1$ a complex Hadamard matrix?

$$H = nI \iff (\sum_{i=0}^{d} \alpha_i \sum_{k=0}^{d} p_{ki} E_k)(\sum_{j=0}^{d} \overline{\alpha_j} \sum_{k=0}^{d} p_{kj} E_k) = nI$$

$$\iff \sum_{k=0}^{d} \sum_{i=0}^{d} \sum_{j=0}^{d} \alpha_i \overline{\alpha_j} p_{ki} p_{kj} E_k = nI$$

$$\iff$$

$$\iff$$

$$A_i = \sum_{k=0}^{d} p_{ki} E_k, \quad E_j E_k = \delta_{jk} E_j, \quad p_{ki} \in \mathbb{R}.$$

When is $H = \sum_{i=0}^{d} \alpha_i A_i$ with $|\alpha_i| = 1$ a complex Hadamard matrix?

$$H = nI \iff (\sum_{i=0}^{d} \alpha_i \sum_{k=0}^{d} p_{ki} E_k)(\sum_{j=0}^{d} \overline{\alpha_j} \sum_{k=0}^{d} p_{kj} E_k) = nI$$

$$\iff \sum_{k=0}^{d} \sum_{i=0}^{d} \sum_{j=0}^{d} \alpha_i \overline{\alpha_j} p_{ki} p_{kj} E_k = nI = n \sum_{k=0}^{d} E_k$$

$$\iff$$

$$\iff$$

$$A_i = \sum_{k=0}^{d} p_{ki} E_k, \quad E_j E_k = \delta_{jk} E_j, \quad p_{ki} \in \mathbb{R}.$$

When is $H = \sum_{i=0}^{d} \alpha_i A_i$ with $|\alpha_i| = 1$ a complex Hadamard matrix?

$$H = nI \iff \Big(\sum_{i=0}^{d} \alpha_i \sum_{k=0}^{d} p_{ki} E_k\Big)\Big(\sum_{j=0}^{d} \overline{\alpha_j} \sum_{k=0}^{d} p_{kj} E_k\Big) = nI$$

$$\iff \sum_{k=0}^{d} \sum_{i=0}^{d} \sum_{j=0}^{d} \alpha_i \overline{\alpha_j} p_{ki} p_{kj} E_k = nI = n \sum_{k=0}^{d} E_k$$

$$\iff \sum_{i=0}^{d} \sum_{j=0}^{d} \alpha_i \overline{\alpha_j} p_{ki} p_{kj} = n \quad (\forall k)$$

$$\iff$$

$$A_i = \sum_{k=0}^{d} p_{ki} E_k, \quad E_j E_k = \delta_{jk} E_j, \quad p_{ki} \in \mathbb{R}.$$

When is $H = \sum_{i=0}^{d} \alpha_i A_i$ with $|\alpha_i| = 1$ a complex Hadamard matrix?

$$H = nI \iff \left(\sum_{i=0}^{d} \alpha_i \sum_{k=0}^{d} p_{ki} E_k\right)\left(\sum_{j=0}^{d} \overline{\alpha_j} \sum_{k=0}^{d} p_{kj} E_k\right) = nI$$

$$\iff \sum_{k=0}^{d} \sum_{i=0}^{d} \sum_{j=0}^{d} \alpha_i \overline{\alpha_j} p_{ki} p_{kj} E_k = nI = n \sum_{k=0}^{d} E_k$$

$$\iff \sum_{i=0}^{d} \sum_{j=0}^{d} \frac{\alpha_i}{\alpha_j} p_{ki} p_{kj} = n \quad (\forall k)$$

$$\iff$$

$$A_i = \sum_{k=0}^{d} p_{ki} E_k, \quad E_j E_k = \delta_{jk} E_j, \quad p_{ki} \in \mathbb{R}.$$

When is $H = \sum_{i=0}^{d} \alpha_i A_i$ with $|\alpha_i| = 1$ a complex Hadamard matrix?

$$H = nI \iff (\sum_{i=0}^{d} \alpha_i \sum_{k=0}^{d} p_{ki} E_k)(\sum_{j=0}^{d} \overline{\alpha_j} \sum_{k=0}^{d} p_{kj} E_k) = nI$$

$$\iff \sum_{k=0}^{d} \sum_{i=0}^{d} \sum_{j=0}^{d} \alpha_i \overline{\alpha_j} p_{ki} p_{kj} E_k = nI = n \sum_{k=0}^{d} E_k$$

$$\iff \sum_{i=0}^{d} \sum_{j=0}^{d} \frac{\alpha_i}{\alpha_j} p_{ki} p_{kj} = n \quad (\forall k)$$

$$\iff \sum_{0 \le i < j \le d} \left(\frac{\alpha_i}{\alpha_j} + \frac{\alpha_j}{\alpha_i} \right) p_{ki} p_{kj} = n - \sum_{i=0}^{d} p_{ki}^2 \quad (\forall k)$$

$$a_{ij} = \frac{\alpha_i}{\alpha_j} + \frac{\alpha_j}{\alpha_i} \quad (0 \leq i < j \leq d). \tag{1}$$

$$\sum_{0 \leq i < j \leq d} a_{ij} p_{ki} p_{kj} = n - \sum_{i=0}^{d} p_{ki}^2 \quad (\forall k) \tag{2}$$

$$a_{ij} = \frac{\alpha_i}{\alpha_j} + \frac{\alpha_j}{\alpha_i} \quad (0 \le i < j \le d). \tag{1}$$

$$\sum_{0 \le i < j \le d} a_{ij} p_{ki} p_{kj} = n - \sum_{i=0}^{d} p_{ki}^2 \quad (\forall k) \tag{2}$$

Step 1 Solve the system of linear equations (2) in $\{a_{ij}\}$

Step 2 Find $\{\alpha_i\}$ from $\{a_{ij}\}$ by (1).

$$a_{ij} = \frac{\alpha_i}{\alpha_j} + \frac{\alpha_j}{\alpha_i} \quad (0 \leq i < j \leq d). \tag{1}$$

$$\sum_{0 \leq i < j \leq d} a_{ij} p_{ki} p_{kj} = n - \sum_{i=0}^{d} p_{ki}^2 \quad (\forall k) \tag{2}$$

Step 1 Solve the system of linear equations (2) in $\{a_{ij}\}$

Step 2 Find $\{\alpha_i\}$ from $\{a_{ij}\}$ by (1).

Problem: Given $\{a_{ij}\}$, when $\exists \{\alpha_i\}$ satisfying (1)?

$$a_{ij} = \frac{\alpha_i}{\alpha_j} + \frac{\alpha_j}{\alpha_i} \quad (0 \le i < j \le d). \tag{1}$$

$$\sum_{0\le i<j\le d} a_{ij} p_{ki} p_{kj} = n - \sum_{i=0}^{d} p_{ki}^2 \quad (\forall k) \tag{2}$$

Step 1 Solve the system of linear equations (2) in $\{a_{ij}\}$

Step 2 Find $\{\alpha_i\}$ from $\{a_{ij}\}$ by (1).

Problem: Given $\{a_{ij}\}$, when $\exists\{\alpha_i\}$ satisfying (1)?

$$\begin{array}{rcl} f : (S^1)^{d+1} & \to & \mathbb{R}^{d(d+1)/2}, \\ \{\alpha_i\}_{i=0}^{d} & \mapsto & \{\frac{\alpha_i}{\alpha_j} + \frac{\alpha_j}{\alpha_i}\}_{0\le i<j<d}. \end{array}$$

where $S^1 = \{\zeta \in \mathbb{C} \mid |\zeta| = 1\}$.

$$a_{ij} = \frac{\alpha_i}{\alpha_j} + \frac{\alpha_j}{\alpha_i} \quad (0 \le i < j \le d). \tag{1}$$

$$\sum_{0 \le i < j \le d} a_{ij} p_{ki} p_{kj} = n - \sum_{i=0}^{d} p_{ki}^2 \quad (\forall k) \tag{2}$$

Step 1 Solve the system of linear equations (2) in $\{a_{ij}\}$

Step 2 Find $\{\alpha_i\}$ from $\{a_{ij}\}$ by (1).

Problem: Given $\{a_{ij}\}$, when $\exists \{\alpha_i\}$ satisfying (1)?

$$\begin{array}{rcl} f : (S^1)^{d+1} & \longrightarrow & \mathbb{R}^{d(d+1)/2}, \\ \{\alpha_i\}_{i=0}^{d} & \mapsto & \{\frac{\alpha_i}{\alpha_j} + \frac{\alpha_j}{\alpha_i}\}_{0 \le i < j < d}. \end{array}$$

where $S^1 = \{\zeta \in \mathbb{C} \mid |\zeta| = 1\}$. Describe image of f

Instead of considering $f : \left(S^1\right)^{d+1} \longrightarrow \mathbb{R}^{d(d+1)/2}$, consider

$$\begin{array}{rcl} f : \left(\mathbb{C}^\times\right)^{d+1} & \longrightarrow & \mathbb{C}^{d(d+1)/2}, \\ \{\alpha_i\}_{i=0}^d & \mapsto & \left\{\frac{\alpha_i}{\alpha_j} + \frac{\alpha_j}{\alpha_i}\right\}_{0\leq i<j<d}. \end{array}$$

Describe the image of f. For example, for $d = 2$:

Instead of considering $f : (S^1)^{d+1} \to \mathbb{R}^{d(d+1)/2}$, consider

$$\begin{aligned} f : (\mathbb{C}^\times)^{d+1} &\longrightarrow \mathbb{C}^{d(d+1)/2}, \\ \{\alpha_i\}_{i=0}^d &\longmapsto \left\{\tfrac{\alpha_i}{\alpha_j} + \tfrac{\alpha_j}{\alpha_i}\right\}_{0\le i<j<d}. \end{aligned}$$

Describe the image of f. For example, for $d = 2$:

$$\begin{aligned} f : (\mathbb{C}^\times)^3 &\longrightarrow \mathbb{C}^3, \\ (x, y, z) &\longmapsto \left(\tfrac{x}{y} + \tfrac{y}{x}, \tfrac{x}{z} + \tfrac{z}{x}, \tfrac{y}{z} + \tfrac{z}{y}\right) \end{aligned}$$

Instead of considering $f : (S^1)^{d+1} \to \mathbb{R}^{d(d+1)/2}$, consider

$$\begin{aligned} f : (\mathbb{C}^\times)^{d+1} &\longrightarrow \mathbb{C}^{d(d+1)/2}, \\ \{\alpha_i\}_{i=0}^{d} &\longmapsto \left\{\tfrac{\alpha_i}{\alpha_j} + \tfrac{\alpha_j}{\alpha_i}\right\}_{0 \leq i < j < d}. \end{aligned}$$

Describe the image of f. For example, for $d = 2$:

$$\begin{aligned} f : (\mathbb{C}^\times)^3 &\longrightarrow \mathbb{C}^3, \\ (x, y, z) &\longmapsto \left(\tfrac{x}{y} + \tfrac{y}{x}, \tfrac{x}{z} + \tfrac{z}{x}, \tfrac{y}{z} + \tfrac{z}{y}\right) \end{aligned}$$

Not surjective.

Instead of considering $f : (S^1)^{d+1} \longrightarrow \mathbb{R}^{d(d+1)/2}$, consider

$$\begin{array}{rcl} f : (\mathbb{C}^\times)^{d+1} & \longrightarrow & \mathbb{C}^{d(d+1)/2}, \\ \{\alpha_i\}_{i=0}^{d} & \longmapsto & \{\frac{\alpha_i}{\alpha_j} + \frac{\alpha_j}{\alpha_i}\}_{0 \leq i < j < d}. \end{array}$$

Describe the image of f. For example, for $d = 2$:

$$\begin{array}{rcl} f : (\mathbb{C}^\times)^3 & \longrightarrow & \mathbb{C}^3, \\ (x, y, z) & \longmapsto & (\frac{x}{y} + \frac{y}{x}, \frac{x}{z} + \frac{z}{x}, \frac{y}{z} + \frac{z}{y}) \end{array}$$

Not surjective. $g(X, Y, Z) = X^2 + Y^2 + Z^2 - XYZ - 4$.

$$g(\frac{x}{y} + \frac{y}{x}, \frac{x}{z} + \frac{z}{x}, \frac{y}{z} + \frac{z}{y}) = 0$$

Instead of considering $f : (S^1)^{d+1} \longrightarrow \mathbb{R}^{d(d+1)/2}$, consider

$$\begin{array}{rcl} f : (\mathbb{C}^\times)^{d+1} & \longrightarrow & \mathbb{C}^{d(d+1)/2}, \\ \{\alpha_i\}_{i=0}^d & \longmapsto & \{\frac{\alpha_i}{\alpha_j} + \frac{\alpha_j}{\alpha_i}\}_{0 \le i < j < d}. \end{array}$$

Describe the image of f. For example, for $d = 2$:

$$\begin{array}{rcl} f : (\mathbb{C}^\times)^3 & \longrightarrow & \mathbb{C}^3, \\ (x, y, z) & \longmapsto & (\frac{x}{y} + \frac{y}{x}, \frac{x}{z} + \frac{z}{x}, \frac{y}{z} + \frac{z}{y}) \end{array}$$

Not surjective. $g(X, Y, Z) = X^2 + Y^2 + Z^2 - XYZ - 4$.

$$g(\frac{x}{y} + \frac{y}{x}, \frac{x}{z} + \frac{z}{x}, \frac{y}{z} + \frac{z}{y}) = 0$$

Indeed, image of f $=$ zeros of g.

$$\begin{aligned} f : (\mathbb{C}^\times)^4 &\longrightarrow \mathbb{C}^6, \\ (x_0, x_1, x_2, x_3) &\longmapsto \left(\tfrac{x_i}{x_j} + \tfrac{x_j}{x_i}\right)_{0 \le i < j \le 3} \end{aligned}$$

$$\begin{aligned} f : (\mathbb{C}^\times)^4 &\longrightarrow \mathbb{C}^6, \\ (x_0, x_1, x_2, x_3) &\longmapsto \left(\tfrac{x_i}{x_j} + \tfrac{x_j}{x_i}\right)_{0\le i<j\le 3} \end{aligned}$$

$$g(X, Y, Z) = X^2 + Y^2 + Z^2 - XYZ - 4.$$

$$g_{i,j,k} = g\left(\frac{x_i}{x_j} + \frac{x_j}{x_i}, \frac{x_i}{x_k} + \frac{x_k}{x_i}, \frac{x_j}{x_k} + \frac{x_k}{x_j}\right) = 0$$

$$
\begin{aligned}
f : (\mathbb{C}^{\times})^{4} &\longrightarrow \mathbb{C}^6, \\
(x_0, x_1, x_2, x_3) &\longmapsto \left(\tfrac{x_i}{x_j} + \tfrac{x_j}{x_i}\right)_{0 \le i < j \le 3}
\end{aligned}
$$

$g(X, Y, Z) = X^2 + Y^2 + Z^2 - XYZ - 4$.

$$
g_{i,j,k} = g\left(\frac{x_i}{x_j} + \frac{x_j}{x_i}, \frac{x_i}{x_k} + \frac{x_k}{x_i}, \frac{x_j}{x_k} + \frac{x_k}{x_j}\right) = 0
$$

image of $f \stackrel{?}{=}$ zeros of $\{g_{i,j,k}\}$.

$$\begin{aligned} f : (\mathbb{C}^{\times})^{4} &\longrightarrow \mathbb{C}^{6}, \\ (x_0, x_1, x_2, x_3) &\longmapsto \left(\tfrac{x_i}{x_j} + \tfrac{x_j}{x_i}\right)_{0 \le i < j \le 3} \end{aligned}$$

$g(X, Y, Z) = X^2 + Y^2 + Z^2 - XYZ - 4.$

$$g_{i,j,k} = g\left(\frac{x_i}{x_j} + \frac{x_j}{x_i}, \frac{x_i}{x_k} + \frac{x_k}{x_i}, \frac{x_j}{x_k} + \frac{x_k}{x_j}\right) = 0$$

image of $f \neq$ zeros of $\{g_{i,j,k}\}$.

$$\begin{array}{rcl} f : (\mathbb{C}^\times)^4 & \longrightarrow & \mathbb{C}^6, \\ (x_0, x_1, x_2, x_3) & \mapsto & (\frac{x_i}{x_j} + \frac{x_j}{x_i})_{0 \le i < j \le 3} \end{array}$$

$g(X, Y, Z) = X^2 + Y^2 + Z^2 - XYZ - 4$.

$$g_{i,j,k} = g(\frac{x_i}{x_j} + \frac{x_j}{x_i}, \frac{x_i}{x_k} + \frac{x_k}{x_i}, \frac{x_j}{x_k} + \frac{x_k}{x_j}) = 0$$

image of $f \neq$ zeros of $\{g_{i,j,k}\}$. Need

$$\begin{aligned} h = {} & (X_{03}^2 - 4)X_{12} - X_{03}(X_{01}X_{23} + X_{02}X_{13}) \\ & + 2(X_{01}X_{02} + X_{13}X_{23}) \qquad (X_{ij} = \frac{x_i}{x_j} + \frac{x_j}{x_i}) \end{aligned}$$

(and similar polynomials obtained by permuting indices)

$$\begin{array}{rcl} f : (\mathbb{C}^\times)^4 & \longrightarrow & \mathbb{C}^6, \\ (x_0, x_1, x_2, x_3) & \longmapsto & (\frac{x_i}{x_j} + \frac{x_j}{x_i})_{0 \leq i < j \leq 3} \end{array}$$

$g(X, Y, Z) = X^2 + Y^2 + Z^2 - XYZ - 4$.

$$g_{i,j,k} = g(\frac{x_i}{x_j} + \frac{x_j}{x_i}, \frac{x_i}{x_k} + \frac{x_k}{x_i}, \frac{x_j}{x_k} + \frac{x_k}{x_j}) = 0$$

image of $f \neq$ zeros of $\{g_{i,j,k}\}$. Need

$$\begin{aligned} h = & (X_{03}^2 - 4)X_{12} - X_{03}(X_{01}X_{23} + X_{02}X_{13}) \\ & + 2(X_{01}X_{02} + X_{13}X_{23}) \qquad (X_{ij} = \frac{x_i}{x_j} + \frac{x_j}{x_i}) \end{aligned}$$

(and similar polynomials obtained by permuting indices)
image of $f =$ zeros of g, h.

$$\begin{aligned} f : (\mathbb{C}^\times)^4 &\to \mathbb{C}^6, \\ (x_0, x_1, x_2, x_3) &\mapsto \left(\tfrac{x_i}{x_j} + \tfrac{x_j}{x_i}\right)_{0 \le i < j \le 3} \end{aligned}$$

$g(X, Y, Z) = X^2 + Y^2 + Z^2 - XYZ - 4$.

$$g_{i,j,k} = g\left(\frac{x_i}{x_j} + \frac{x_j}{x_i}, \frac{x_i}{x_k} + \frac{x_k}{x_i}, \frac{x_j}{x_k} + \frac{x_k}{x_j}\right) = 0$$

image of $f \neq$ zeros of $\{g_{i,j,k}\}$. Need

$$\begin{aligned} h = (X_{03}^2 - 4)X_{12} - X_{03}(X_{01}X_{23} + X_{02}X_{13}) \\ + 2(X_{01}X_{02} + X_{13}X_{23}) \qquad \left(X_{ij} = \frac{x_i}{x_j} + \frac{x_j}{x_i}\right) \end{aligned}$$

(and similar polynomials obtained by permuting indices)
image of $f =$ zeros of g, h.
The same is true for $\forall m \geq 4$.

Theorem

$$\begin{array}{rcl} f:(\mathbb{C}^\times)^{d+1} & \longrightarrow & \mathbb{C}^{d(d+1)/2}, \\ (\alpha_0,\alpha_1,\ldots,\alpha_d) & \mapsto & (\frac{\alpha_i}{\alpha_j}+\frac{\alpha_j}{\alpha_i})_{0\le i<j\le d} \end{array}$$

The image of f coincides with the set of zeros of the ideal I in the polynomial ring $\mathbb{C}[X_{ij} : 0 \le i < j \le d]$ generated by

$$g(X_{ij}, X_{ik}, X_{jk})$$
$$h(X_{ij}, X_{ik}, X_{il}, X_{jk}, X_{jl}, X_{kl})$$

where i, j, k, l are distinct, $X_{ij} = X_{ji}$, and

$$g = X^2 + Y^2 + Z^2 - XYZ - 4,$$
$$h = (Z^2 - 4)U - Z(XW + YV) + 2(XY + VW).$$

Given a zero (a_{ij}) of the ideal I, we know that there exists $(\alpha_i) \in (\mathbb{C}^\times)^{d+1}$ such that

$$a_{ij} = \frac{\alpha_i}{\alpha_j} + \frac{\alpha_j}{\alpha_i} \quad (0 \leq i < j \leq d). \tag{1}$$

Given a zero (a_{ij}) of the ideal I, we know that there exists $(\alpha_i) \in (\mathbb{C}^\times)^{d+1}$ such that

$$a_{ij} = \frac{\alpha_i}{\alpha_j} + \frac{\alpha_j}{\alpha_i} \quad (0 \le i < j \le d). \tag{1}$$

How do we find (α_i), and when does $(\alpha_i) \in (S^1)^{d+1}$ hold?

Given a zero (a_{ij}) of the ideal I, we know that there exists $(\alpha_i) \in (\mathbb{C}^\times)^{d+1}$ such that

$$a_{ij} = \frac{\alpha_i}{\alpha_j} + \frac{\alpha_j}{\alpha_i} \quad (0 \le i < j \le d). \tag{1}$$

How do we find (α_i), and when does $(\alpha_i) \in (S^1)^{d+1}$ hold? Observe, for $\alpha \in \mathbb{C}^\times$,

$$|\alpha| = 1 \iff -2 \le \alpha + \frac{1}{\alpha} \le 2.$$

Given a zero (a_{ij}) of the ideal I, we know that there exists $(\alpha_i) \in (\mathbb{C}^\times)^{d+1}$ such that

$$a_{ij} = \frac{\alpha_i}{\alpha_j} + \frac{\alpha_j}{\alpha_i} \quad (0 \le i < j \le d). \tag{1}$$

How do we find (α_i), and when does $(\alpha_i) \in (S^1)^{d+1}$ hold? Observe, for $\alpha \in \mathbb{C}^\times$,

$$|\alpha| = 1 \iff -2 \le \alpha + \frac{1}{\alpha} \le 2.$$

Given a zero (a_{ij}) of the ideal I, we know that there exists $(\alpha_i) \in (\mathbb{C}^\times)^{d+1}$ such that

$$a_{ij} = \frac{\alpha_i}{\alpha_j} + \frac{\alpha_j}{\alpha_i} \quad (0 \le i < j \le d). \tag{1}$$

How do we find (α_i), and when does $(\alpha_i) \in (S^1)^{d+1}$ hold? Observe, for $\alpha \in \mathbb{C}^\times$,

$$|\alpha| = 1 \iff -2 \le \alpha + \frac{1}{\alpha} \le 2.$$

So we need $-2 \le a_{ij} \le 2$.

Given a zero (a_{ij}) of the ideal I, we know that there exists $(\alpha_i) \in (\mathbb{C}^\times)^{d+1}$ such that

$$a_{ij} = \frac{\alpha_i}{\alpha_j} + \frac{\alpha_j}{\alpha_i} \quad (0 \le i < j \le d). \tag{1}$$

How do we find (α_i), and when does $(\alpha_i) \in (S^1)^{d+1}$ hold?
Observe, for $\alpha \in \mathbb{C}^\times$,

$$|\alpha| = 1 \iff -2 \le \alpha + \frac{1}{\alpha} \le 2.$$

So we need $-2 \le a_{ij} \le 2$.
Moreover, if $a_{ij} \in \{\pm 2\}$ for all i, j, then $\alpha_i = \pm\alpha_j$ so the resulting matrix is a scalar multiple of a real Hadamard matrix $\rightarrow$ Goethals–Seidel (1970).

Theorem

$$\begin{array}{rcl} f:(\mathbb{C}^\times)^{d+1} & \to & \mathbb{C}^{d(d+1)/2}, \\ (\alpha_0,\alpha_1,\ldots,\alpha_d) & \mapsto & \left(\frac{\alpha_i}{\alpha_j}+\frac{\alpha_j}{\alpha_i}\right)_{0\le i<j\le d} \end{array}$$

Suppose $(a_{ij}) \in$ the image of f, $a_{ij} \in \mathbb{R}$, and there exists $0 \le i_0 < i_1 \le d$ such that $-2 < a_{i_0,i_1} < 2$. Let $\alpha_{i_0}, \alpha_{i_1}$ be

$$a_{i_0,i_1} = \frac{\alpha_{i_0}}{\alpha_{i_1}} + \frac{\alpha_{i_1}}{\alpha_{i_0}}$$

Define α_i $(0 \le i \le n,\ i \ne i_0, i_1)$ by

$$\alpha_i = \frac{\alpha_{i_0}(a_{i_0,i_1}\alpha_{i_1} - 2\alpha_{i_0})}{a_{i_1,i}\alpha_{i_1} - a_{i_0,i}\alpha_{i_0}}.$$

Then $|\alpha_i| = |\alpha_j|$ and

$$\frac{\alpha_i}{\alpha_j} + \frac{\alpha_j}{\alpha_i} = a_{ij} \quad (0 \le i < j \le d). \tag{1}$$

and every (α_i) satisfying (1) is obtained in this way.

Theorem

$$\begin{aligned} f : (\mathbb{C}^\times)^{d+1} &\to \mathbb{C}^{d(d+1)/2}, \\ (\alpha_0, \alpha_1, \ldots, \alpha_d) &\mapsto \left(\tfrac{\alpha_i}{\alpha_j} + \tfrac{\alpha_j}{\alpha_i}\right)_{0 \le i < j \le d} \end{aligned}$$

Suppose $(a_{ij}) \in$ the image of f, $a_{ij} \in \mathbb{R}$, and there exists $0 \le i_0 < i_1 \le d$ such that $-2 < a_{i_0,i_1} < 2$. Let $\alpha_{i_0}, \alpha_{i_1}$ be

$$a_{i_0,i_1} = \frac{\alpha_{i_0}}{\alpha_{i_1}} + \frac{\alpha_{i_1}}{\alpha_{i_0}} = \frac{\alpha_{i_0}}{\alpha_{i_1}} + \left(\frac{\alpha_{i_0}}{\alpha_{i_1}}\right)^{-1}$$

Define α_i $(0 \le i \le n,\ i \ne i_0, i_1)$ by

$$\alpha_i = \frac{\alpha_{i_0}(a_{i_0,i_1}\alpha_{i_1} - 2\alpha_{i_0})}{a_{i_1,i}\alpha_{i_1} - a_{i_0,i}\alpha_{i_0}}.$$

Then $|\alpha_i| = |\alpha_j|$ and

$$\frac{\alpha_i}{\alpha_j} + \frac{\alpha_j}{\alpha_i} = a_{ij} \quad (0 \le i < j \le d). \tag{1}$$

and every (α_i) satisfying (1) is obtained in this way.

The procedure

Step 1 Solve the system of equations

$$g(X_{ij}, X_{ik}, X_{jk}) = 0,$$
$$h(X_{ij}, X_{ik}, X_{il}, X_{jk}, X_{jl}, X_{kl}) = 0,$$
$$\sum_{0 \le i < j \le d} X_{ij} p_{ki} p_{kj} = n - \sum_{i=0}^{d} p_{ki}^2$$

Step 2 List all solutions a_{ij} with $-2 \le a_{ij} \le 2$.

Step 3 Find (α_i) by

$$\alpha_i = \frac{\alpha_{i_0}(a_{i_0,i_1}\alpha_{i_1} - 2\alpha_{i_0})}{a_{i_1,i}\alpha_{i_1} - a_{i_0,i}\alpha_{i_0}}.$$

where $a_{i_0,i_1} \neq \pm 2$,

$$\frac{\alpha_{i_0}}{\alpha_{i_1}} + \frac{\alpha_{i_1}}{\alpha_{i_0}} = a_{i_0,i_1}.$$

- Step 1 Solve the system of equations
- Step 2 List all solutions a_{ij} with $-2 \leq a_{ij} \leq 2$,
- Step 3 Find (α_i)

Step 1 Solve the system of equations

Step 2 List all solutions a_{ij} with $-2 \leq a_{ij} \leq 2$,

Step 3 Find (α_i)

In many known examples of association schemes with $d = 3$, Step 2 failed.

Step 1 Solve the system of equations

Step 2 List all solutions a_{ij} with $-2 \leq a_{ij} \leq 2$,

Step 3 Find (α_i)

In many known examples of association schemes with $d = 3$, Step 2 failed.

Theorem (Chan, arXiv:1102.5601v1)

There are only finitely many antipodal distance-regular graphs of diameter 3 whose Bose–Mesner algebra contains a complex Hadamard matrix.

Step 1 Solve the system of equations

Step 2 List all solutions a_{ij} with $-2 \leq a_{ij} \leq 2$,

Step 3 Find (α_i)

In many known examples of association schemes with $d = 3$, Step 2 failed.

Theorem (Chan, arXiv:1102.5601v1)

There are only finitely many antipodal distance-regular graphs of diameter 3 whose Bose–Mesner algebra contains a complex Hadamard matrix.

But Chan did find an example. $L(O_3)$: the line graph of the Petersen graph.

- q: a power of 2, $q \geq 4$,
- $\Omega = \mathrm{PG}(2, q)$: the projective plane over $\mathbb{F}_q$,
- $Q = \{[a_0, a_1, a_2] \in \Omega \mid a_0^2 + a_1 a_2 = 0\}$: quadric,
- $X = \{[a_0, a_1, a_2] \in \Omega \setminus Q \mid [a_0, a_1, a_2] \neq [1, 0, 0]\}$,
- $|X| = q^2 - 1$.

- q: a power of 2, $q \geq 4$,
- $\Omega = \mathrm{PG}(2, q)$: the projective plane over $\mathbb{F}_q$,
- $Q = \{[a_0, a_1, a_2] \in \Omega \mid a_0^2 + a_1 a_2 = 0\}$: quadric,
- $X = \{[a_0, a_1, a_2] \in \Omega \setminus Q \mid [a_0, a_1, a_2] \neq [1, 0, 0]\}$,
- $|X| = q^2 - 1$.

For $x, y \in X$, denote by $x + y$ the line through x, y.

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } i = 0,\ x = y, \\ 1 & \text{if } i = 1,\ |(x+y) \cap Q| = 2, \\ 1 & \text{if } i = 2,\ |(x+y) \cap Q| = 0, \\ 1 & \text{if } i = 3,\ |(x+y) \cap Q| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

$\exists$ a complex Hadamard matrix in its Bose–Mesner algebra.

- q: a power of 2, $q \geq 4$,
- $\Omega = \mathrm{PG}(2, q)$: the projective plane over $\mathbb{F}_q$,
- $Q = \{[a_0, a_1, a_2] \in \Omega \mid a_0^2 + a_1 a_2 = 0\}$: quadric,
- $X = \{[a_0, a_1, a_2] \in \Omega \setminus Q \mid [a_0, a_1, a_2] \neq [1, 0, 0]\}$,
- $|X| = q^2 - 1$.

For $x, y \in X$, denote by $x + y$ the line through x, y.

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } i = 0,\ x = y, \\ 1 & \text{if } i = 1,\ |(x+y) \cap Q| = 2, \\ 1 & \text{if } i = 2,\ |(x+y) \cap Q| = 0, \\ 1 & \text{if } i = 3,\ |(x+y) \cap Q| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

$\exists$ a complex Hadamard matrix in its Bose–Mesner algebra.

- q: a power of 2, $q \geq 4 \rightarrow q = 4$,
- $\Omega = \mathrm{PG}(2, q)$: the projective plane over $\mathbb{F}_q$,
- $Q = \{[a_0, a_1, a_2] \in \Omega \mid a_0^2 + a_1 a_2 = 0\}$: quadric,
- $X = \{[a_0, a_1, a_2] \in \Omega \setminus Q \mid [a_0, a_1, a_2] \neq [1, 0, 0]\}$,
- $|X| = q^2 - 1 \rightarrow 15$.

For $x, y \in X$, denote by $x + y$ the line through x, y.

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } i = 0,\ x = y, \\ 1 & \text{if } i = 1,\ |(x+y) \cap Q| = 2, \\ 1 & \text{if } i = 2,\ |(x+y) \cap Q| = 0, \\ 1 & \text{if } i = 3,\ |(x+y) \cap Q| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

$\exists$ a complex Hadamard matrix in $L(O_3)$.

Theorem

The matrix $H = I + \alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3$ is a complex Hadamard matrix if and only if

(i) H belongs to the subalgebra forming the Bose–Mesner algebra of a strongly regular graph (precise description omitted, already done by Chan–Godsil),

(ii)
$$\alpha_1 + \frac{1}{\alpha_1} = -\frac{2}{q}, \quad \alpha_2 = \frac{1}{\alpha_1}, \quad \alpha_3 = 1,$$

(iii)
$$\alpha_1 + \frac{1}{\alpha_1} = \frac{(q-1)(q-2) - (q+2)r}{q},$$

where $r = \sqrt{(q-1)(17q-1)} > 0$.

The case (ii) with $q = 4$ was found by Chan.