組合せデザインから得られる線形符号

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t-(v, k, λ) designs

Definition

A t-(v, k, λ) design is a pair $(\mathcal{P}, \mathcal{B})$, where

- $\mathcal{P}$: a finite set of "points",
- $\mathcal{B}$: a collection of k-subsets of $\mathcal{P}$, a member of which is called a "block,"
- $\forall T \subset \mathcal{P}$ with $|T| = t$, there are exactly λ members $B \in \mathcal{B}$ such that $T \subset B$.

Examples:

- 2-$(v, 3, 1)$ design = Steiner triple system
- 2-$(q^2, q, 1)$ design = affine plane of order q

$$t\text{-design} \implies (t-1)\text{-design}$$

More precisely,...

Intersection numbers

$(\mathcal{P}, \mathcal{B})$: t-(v, k, λ) design. Write $\lambda = \lambda_t$,

$$\lambda_{t-1} = |\{B \in \mathcal{B} \mid T' \subset B\}|,$$

where $T' \subset \mathcal{P}$, $|T'| = t - 1$. Then

$$\begin{aligned}
\lambda_{t-1}(k - t + 1) &= \sum_{\substack{B \in \mathcal{B} \\ T' \subset B}} |B \setminus T'| \\
&= |\{(B, x) \in \mathcal{B} \mid T' \cup \{x\} \subset B,\ x \in \mathcal{P} \setminus T'\}| \\
&= \sum_{x \in \mathcal{P} \setminus T'} |\{B \in \mathcal{B} \mid T' \cup \{x\} \subset B\}| \\
&= \sum_{x \in \mathcal{P} \setminus T'} \lambda_t \\
&= \lambda_t(v - t + 1).
\end{aligned}$$

$(\mathcal{P},\mathcal{B})$: t-(v,k,λ) design

Then $(\mathcal{P},\mathcal{B})$: $(t-1)$-(v,k,λ_{t-1}) design, where

$$\lambda_{t-1}=\lambda_t\frac{v-t+1}{k-t+1}.$$

For example,

$$\begin{aligned} 5\text{-}(24,8,1) &\implies 4\text{-}(24,8,5)\\ &\implies 3\text{-}(24,8,21)\\ &\implies 2\text{-}(24,8,77)\\ &\implies 1\text{-}(24,8,253)\\ &\implies 0\text{-}(24,8,759)\\ &\iff |\mathcal{B}|=759. \end{aligned}$$

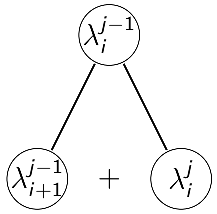
$(\mathcal{P},\mathcal{B})$: t-(v,k,λ) design

Let $I\subset\mathcal{P}$, $J\subset\mathcal{P}$, $|I|=i$, $|J|=j$, $I\cap J=\emptyset$, $i+j\le t$.
Define

$$\lambda_i^j=|\{B\in\mathcal{B}\mid I\subset B,\ B\cap J=\emptyset\}|.$$

In particular,

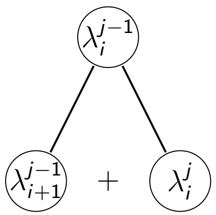
$$\lambda_i^0=\lambda_i\quad(0\le i\le t).$$
$$\lambda_i^{j-1}=\lambda_{i+1}^{j-1}+\lambda_i^j.$$



$$\begin{array}{c} \lambda_0^0\\ \lambda_1^0\ \lambda_0^1\\ \lambda_2^0\ \lambda_1^1\ \lambda_0^2\\ \lambda_3^0\ \lambda_2^1\ \lambda_1^2\ \lambda_0^3\\ \lambda_4^0\ \lambda_3^1\ \lambda_2^2\ \lambda_1^3\ \lambda_0^4\\ \lambda_5^0\ \lambda_4^1\ \lambda_3^2\ \lambda_2^3\ \lambda_1^4\ \lambda_0^5 \end{array}$$

5-$(24,8,1)$ design, $\lambda_i^{j-1}=\lambda_{i+1}^{j-1}+\lambda_i^j$

$$\begin{array}{c} 759\\ 253\quad 506\\ 77\quad 176\quad 330\\ 21\quad 56\quad 120\quad 210\\ 5\quad 16\quad 40\quad 80\quad 130\\ 1\quad 4\quad 12\quad 28\quad 52\quad 78 \end{array}$$



Next row?
λ_6^0, λ_5^1, λ_4^2, . . .

$$\lambda_6^0(I)=|\{B\in\mathcal{B}\mid I\subset B\}|=1\text{ or }0$$

depending on the choice of $I\subset\mathcal{P}$ with $|I|=6$.
Choose I in such a way that $\lambda_6^0(I)=1$.

5-$(24,8,1)$ design, $I\subset\mathcal{P}$, $|I|=6$, $I\subset\exists B\in\mathcal{B}$

$$\lambda_{6-j}^j=|\{B\in\mathcal{B}\mid I\setminus J\subset B,\ B\cap J=\emptyset\}|\quad\text{where } J\subset I,\ J=j.$$

$$\lambda_{5-j}^j=\lambda_{6-j}^j+\lambda_{5-j}^{j+1}$$

giving

$$\begin{array}{c} 759\\ 253\quad 506\\ 77\quad 176\quad 330\\ 21\quad 56\quad 120\quad 210\\ 5\quad 16\quad 40\quad 80\quad 130\\ 1\quad 4\quad 12\quad 28\quad 52\quad 78\\ 1\quad 0\quad 4\quad 8\quad 20\quad 32\quad 46 \end{array}$$

Similarly, taking $I\subset\mathcal{P}$, $|I|=7$ appropriately, we obtain λ_{7-j}^j.
Finally taking $I\in\mathcal{B}$, we obtain λ_{8-j}^j.

5-(24, 8, 1) design

759								
253	506							
77	176	330						
21	56	120	210					
5	16	40	80	130				
1	4	12	28	52	78			
1	0	4	8	20	32	46		
1	0	0	4	4	16	16	30	
1	0	0	0	4	0	16	0	30

The last row implies

$$B, B' \in \mathcal{P},\ B \neq B' \implies |B \cap B'| \in \{4, 2, 0\}.$$

Todd's lemma

Let $(\mathcal{P}, \mathcal{B})$ be a 5-(24, 8, 1) design. Then

$$B, B' \in \mathcal{B},\ |B \cap B'| = 4 \implies B \triangle B' \in \mathcal{B}.$$

Proof by contradiction:

1	2	3	4	5	6	7	8								
1	2	3	4					9	10	11	12				
				5	6	7	8	9	10			13	14		
				5	6	7	8			11	12			15	16
*	*	*	*	5	6	7		9		11					

Here "****" must be odd and even simultaneously.

Binary codes

A (linear) binary code of length v is a subspace of the vector space $\mathbb{F}_2^v$. If C is a binary code and $\dim C = k$, we say C is an binary $[v, k]$ code.
The dual code of a binary code C is defined as

$$C^\perp = \{x \in \mathbb{F}_2^v \mid x \cdot y = 0\ (\forall y \in C)\}.$$

where

$$x \cdot y = \sum_{i=1}^{v} x_i y_i.$$

Then

$$\dim C^\perp = v - \dim C.$$

The code C is said to be self-orthogonal if $C \subset C^\perp$ and self-dual if $C = C^\perp$.

Weight

For $x \in \mathbb{F}_2^v$, we write

$$\mathrm{supp}(x) = \{i \mid 1 \le i \le v,\ x_i \neq 0\},$$
$$\mathrm{wt}(x) = |\,\mathrm{supp}(x)|.$$

For a binary code C, its minimum weight is

$$\min\{\mathrm{wt}(x) \mid 0 \neq x \in C\}.$$

If an $[v, k]$ code C has minimum weight d, we call C an $[v, k, d]$ code. C is doubly even if $\mathrm{wt}(x) \equiv 0 \pmod 4$ $(\forall x \in C)$. Note

$$C \subset C^\perp \iff |\,\mathrm{supp}(x) \cap \mathrm{supp}(y)| \equiv 0 \pmod 2 \quad (\forall x, y \in C).$$

Generator matrix of a code

If a binary code C is generated by row vectors $x^{(1)}, \dots, x^{(b)}$, then the matrix

$$\begin{bmatrix} x^{(1)} \\ \vdots \\ x^{(b)} \end{bmatrix}$$

is called a generator matrix of C. This means

$$C = \{\sum_{i=1}^{b} \epsilon_i x^{(i)} \mid \epsilon_1, \dots, \epsilon_b \in \mathbb{F}_2\} \subset \mathbb{F}_2^v.$$

Note

$$C \subset C^\perp \iff |\operatorname{supp}(x^{(i)}) \cap \operatorname{supp}(x^{(j)})| \equiv 0 \pmod 2 \quad (\forall i, j).$$

$$C : \text{doubly even} \iff C \subset C^\perp \text{ and } \operatorname{wt}(x^{(i)}) \equiv 0 \pmod 4 \quad (\forall i).$$

Incidence matrix of a design

If $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is a t-(v, k, λ) design, the incidence matrix $M(\mathcal{D})$ of $\mathcal{D}$ is $|\mathcal{B}| \times |\mathcal{P}|$ matrix whose rows and columns are indexed by $\mathcal{B}$ and $\mathcal{P}$, respectively, such that its (B, p) entry is 1 if $p \in B$, 0 otherwise. In other words, the row vectors of $M(\mathcal{D})$ are the characteristic vectors of blocks:

$$M(\mathcal{D}) = \begin{bmatrix} x^{(B_1)} \\ \vdots \\ x^{(B_b)} \end{bmatrix} \quad : b \times v \text{ matrix},$$

where $\mathcal{B} = \{B_1, \dots, B_b\}$, and $x^{(B)} \in \mathbb{F}_2^v$ denotes the characteristic vector of B, i.e., $\operatorname{supp}(x^{(B)}) = B$.
The binary code of the design $\mathcal{D}$ is the binary code of length v having $M(\mathcal{D})$ as a generator matrix.

$\dim C \le 12$ for 5-$(24, 8, 1)$ design

Recall that in a 5-$(24, 8, 1)$ design $(\mathcal{P}, \mathcal{B})$,

$$|B \cap B'| \in \{8, 4, 2, 0\} \quad (\forall B, B' \in \mathcal{B}).$$

The binary code C of a 5-$(24, 8, 1)$ design is self-orthogonal. Indeed, the incidence matrix has row vectors $x^{(B)}$ $(B \in \mathcal{B})$, the characteristic vector of the block B. Then

$$x^{(B)} \cdot x^{(B')} = |B \cap B'| \bmod 2 = (8 \text{ or } 4 \text{ or } 2 \text{ or } 0) \bmod 2 = 0.$$

Thus $C \subset C^\perp$, hence

$$\dim C \le \frac{1}{2}(\dim C + \dim C^\perp) \le \frac{24}{2} = 12.$$

The 5-$(24, 8, 1)$ design, $|B \cap B'| \in \{4, 2, 0\}$

$\mathcal{P} = \{1, 2, \dots, 24\}$. We may take $\mathcal{B}$ as:

1	2	3	4	5	6	7	8																
1	2	3	4					9	10	11	12												
1	2	3		5				9				13	14	15									
1	2		4	5				9							16	17	18						
1		3	4	5				9										19	20	21			
	2	3	4	5				9													22	23	24
1	2	3			6			9							16			19			22		
1	2		4		6			9				13							20			23	
1		3	4		6			9					14			17							24
1	2			5	6			9	10											21			24
1		3		5	6			9		11							18					23	

Do we have to find 759 blocks one by one?
No, 12 blocks are sufficient (so one more needed).

Consequence of Todd's lemma

1	2	3	4	5	6	7	8																
1	2	3	4					9	10	11	12												
1	2	3		5				9				13	14	15									
1	2		4	5				9							16	17	18						
1		3	4	5				9										19	20	21			
	2	3	4	5				9													22	23	24
1	2	3			6			9							16			19			22		
1	2		4		6			9				13							20			23	
1		3	4		6			9					14			17							24
1	2			5	6			9	10											21			24
1		3		5	6			9		11							18					23	

By Todd's lemma

$$B_0 = ((B_1 \triangle B_4) \triangle B_7) \triangle (B_5 \triangle B_6) = \{7, 8, 17, 18, 20, 21, 23, 24\} \in \mathcal{B}.$$

One more block for 5-(24, 8, 1) design

1	2	3	4	5	6	7	8																
1	2	3	4					9	10	11	12												
1	2	3		5				9				13	14	15									
1	2		4	5				9							16	17	18						
1		3	4	5				9										19	20	21			
	2	3	4	5				9													22	23	24
1	2	3			6			9							16			19			22		
1	2		4		6			9				13							20			23	
1		3	4		6			9					14			17							24
1	2			5	6			9	10											21			24
1		3		5	6			9		11							18					23	

The above 11 blocks generate a 11-dimensional code C_0. Note the transposition (7 8) leaves C_0 invariant. We know from Todd's lemma $B_0 = \{7, 8, 17, 18, 20, 21, 23, 24\} \in \mathcal{B}$ (but $x^{(B_0)} \in C_0$).
Consider the block containing $\{1, 2, 3, 8, 9\}$. There are two choices: $B = \{1, 2, 3, 8, 9, 17, 21, 23\}$ and $B' = \{1, 2, 3, 8, 9, 18, 20, 24\}$.

One more block for 5-(24, 8, 1) design

We know

$$B_0 = \{7, 8, 17, 18, 20, 21, 23, 24\} \in \mathcal{B}, \quad x^{(B_0)} \in C_0 = C_0^{(7\,8)}.$$

We have either

$$B = \{1, 2, 3, 8, 9, 17, 21, 23\} \in \mathcal{B} \text{ or}$$
$$B' = \{1, 2, 3, 8, 9, 18, 20, 24\} \in \mathcal{B}.$$

But $B'^{(7\,8)} = B \triangle B_0$, so

$$\langle C_0, x^{(B')} \rangle^{(7\,8)} = \langle C_0, x^{(B)} + x^{(B_0)} \rangle = \langle C_0, x^{(B)} \rangle.$$

Therefore, the code generated by the design is unique up to isomorphism. This self-dual ($C = C^{\perp}$) code is known as the extended binary Golay code. Next we show that the code determines the design uniquely.

Mendelsohn equations for t-(v, k, λ) design $(\mathcal{P}, \mathcal{B})$

For $S \subset \mathcal{P}$, let

$$n_i(S) = |\{B \in \mathcal{B} \mid i = |B \cap S|\}|.$$

Then

$$\sum_{i \geq 0} \binom{i}{j} n_i(S) = \lambda_j \binom{|S|}{j} \quad (0 \leq j \leq t).$$

Proof: Count

$$\{(J, B) \mid J \subset S \cap B, \ |J| = j\}$$

in two ways.

$n_i(S) = |\{B \in \mathcal{B} \mid i = |B \cap S|\}|$

Let C be the binary code of the design $(\mathcal{P}, \mathcal{B})$.
Write $n_i(\mathrm{supp}(v)) = n_i(v)$ for $v \in \mathbb{F}_2^v$.

$$\sum_{i \geq 0} \binom{i}{j} n_i(v) = \lambda_j \binom{\mathrm{wt}(v)}{j} \quad (0 \leq j \leq t).$$

If $v \in C^\perp$, then $|B \cap \mathrm{supp}(v)|$ is even, so

$$n_i(v) = |\{B \in \mathcal{B} \mid i = |B \cap \mathrm{supp}(v)|\}| = 0 \quad \text{for } i \text{ odd}.$$

Thus

$$\sum_{\substack{0 \leq i \leq \mathrm{wt}(v) \\ i:\ \text{even}}} \binom{i}{j} n_i(v) = \lambda_j \binom{\mathrm{wt}(v)}{j} \quad (0 \leq j \leq t).$$

$(\mathcal{P}, \mathcal{B})$: 5-$(24, 8, 1)$ design

$$\sum_{\substack{0 \leq i \leq \mathrm{wt}(v) \\ i:\ \text{even}}} \binom{i}{j} n_i(v) = \lambda_j \binom{\mathrm{wt}(v)}{j} \quad (0 \leq j \leq 5).$$

Taking $v \in C^\perp$ with $0 < \mathrm{wt}(v) < 8$ gives no solution. This means that $C^\perp$ has minimum weight 8.
Take $v \in C = C^\perp$ with $\mathrm{wt}(v) = 8$. Then there are six equations for five unknowns n_0, n_2, n_4, n_6, n_8. The unique solution is

$$(n_0, n_2, n_4, n_6, n_8) = (30, 448, 280, 0, 1).$$

This implies $\mathrm{supp}(v) \in \mathcal{B}$. Thus

$$\mathcal{B} = \{\mathrm{supp}(x) \mid x \in C,\ \mathrm{wt}(x) = 8\}.$$

Now the uniqueness of the design follows from that of C.

Summary

$\mathcal{D}$: 5-$(24, 8, 1)$ design (Witt system).

- The binary code C of $\mathcal{D}$ is a doubly even self-dual $[24, 12, 8]$ code.
- The binary code C of $\mathcal{D}$ is unique up to isomorphism.
- $\{\mathrm{supp}(x) \mid x \in C,\ \mathrm{wt}(x) = 8\} = \mathcal{B}$.
- There is a unique 5-$(24, 8, 1)$ design up to isomorphism.

The Assmus–Mattson theorem implies that every binary doubly even self-dual $[24, 12, 8]$ code coincides with the binary code of a 5-$(24, 8, 1)$ design, and hence such a code (the extended binary Golay code) is also unique.

The Assmus–Mattson theorem

Theorem

Let C be a binary code of length v, minimum weight k.

$$\begin{aligned} \mathcal{P} &= \{1, 2, \ldots, v\}, \\ \mathcal{B} &= \{\mathrm{supp}(x) \mid x \in C,\ \mathrm{wt}(x) = k\}, \\ S &= \{\mathrm{wt}(x) \mid x \in C^\perp,\ 0 < \mathrm{wt}(x) < v\}, \\ t &= k - |S|. \end{aligned}$$

Then $(\mathcal{P}, \mathcal{B})$ is a t-(v, k, λ) design for some λ.

- C: $[24, 12, 8]$ binary doubly even self-dual ($C = C^\perp$) code, so $k = 8$ and C has only weights $0, 8, 12, 16, 24$.

$$\begin{aligned} S &= \{\mathrm{wt}(x) \mid x \in C^\perp,\ 0 < \mathrm{wt}(x) < 24\} = \{8, 12, 16\}, \\ t &= k - |S| = 8 - 3 = 5. \end{aligned}$$

Uniqueness of the extended binary Golay code

C: $[24, 12, 8]$ binary doubly even self-dual ($C = C^\perp$) code.

- The Assmus–Mattson theorem implies $(\mathcal{P}, \mathcal{B})$ is a 5-$(24, 8, \lambda)$ design, where $\mathcal{P} = \{1, 2, \ldots, 24\}$,

$$\mathcal{B} = \{\text{supp}(x) \mid x \in C,\ \text{wt}(x) = 8\},$$

  for some λ.
- If $\lambda > 1$, then $\exists B, B' \in \mathcal{B}$, $B \neq B'$, $|B \cap B'| \geq 5$. Then $\text{wt}(x^{(B)} + x^{(B')}) < 8$, a contradiction. Thus $\lambda = 1$.
- So C is the binary code of a 5-$(24, 8, 1)$ design which was already shown to be unqiue.

This proves the uniqueness of the extended binary Golay code.

Applicability of the Assmus–Mattson theorem

Theorem

Let C be a binary code of length v, minimum weight k.

$$\mathcal{P} = \{1, 2, \ldots, v\},$$
$$\mathcal{B} = \{\text{supp}(x) \mid x \in C,\ \text{wt}(x) = k\},$$
$$S = \{\text{wt}(x) \mid x \in C^\perp,\ 0 < \text{wt}(x) < v\},$$
$$t = k - |S|.$$

Then $(\mathcal{P}, \mathcal{B})$ is a t-(v, k, λ) design for some λ.

The conclusion is stronger if k is large and $|S|$ is small. These are conflicting requirments:

larger k $\Longrightarrow$ smaller C $\Longrightarrow$ larger $C^\perp$ $\Longrightarrow$ larger S

suppose $C = C^\perp$, doubly even $\Longrightarrow$ S not too large

Binary doubly even self-dual codes

Under what circumstance can one obtain a 5-design from a doubly even self-dual code? Let k be the minimum weight.

$$S = \{\text{wt}(x) \mid x \in C,\ 0 < \text{wt}(x) < v\},$$
$$5 = k - |S|.$$

- $k = 8$, $|S| = 3$, $S = \{8, 12, 16\}$, $v = 24$.
- $k = 12$, $|S| = 7$, $S = \{12, 16, 20, 24, 28, 32, 36\}$, $v = 48$.
- $k = 16$, $|S| = 11$, $S = \{16, 20, 24, 28, 32, 36, 40, 44, 48, 52, 56\}$, $v = 72$.

In general, $\forall k$: a multiple of 4, $|S| = k - 5$,

$$S = \{k, k + 4, k + 8, \ldots, 5k - 24 = v - k\}$$

$v = 6k - 24 = 24m$, where $k = 4m + 4$.

Extremal binary doubly even self-dual codes

Theorem (Mallows–Sloane, 1973)

For $m \geq 1$, a binary doubly even self-dual $[24m, 12m]$ code has minimum weight at most $4m + 4$.

Definition

A binary doubly even self-dual $[24m, 12m]$ code with minimum weight $4m + 4$ is called extremal.

For $m \geq 1$, an extremal binary doubly even self-dual code gives a 5-$(24m, 4m + 4, \lambda)$ design by the Assmus–Mattson theorem.

- $m = 1$: the extended binary Golay code and the 5-$(24, 8, 1)$ design
- $m = 2$: Houghten–Lam–Thiel–Parker (2003): unique $[48, 24, 12]$ code and a 5-$(48, 12, 8)$ design which is unique under self-orthogonality.

Extremal binary doubly even self-dual codes

Definition

A binary doubly even self-dual $[24m, 12m]$ code with minimum weight $4m + 4$ is called extremal.

- For $m \geq 3$, neither a code nor a design is known.

Theorem (Zhang, 1999)

There does not exist an extremal $[24m, 12m, 4m + 4]$ binary doubly even self-dual code for $m \geq 154$.