Complementary Ramsey Numbers

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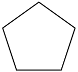
Ramsey Numbers

For a graph G,

$\alpha(G)$ = independence number = $\max\{\#$independent set$\}$
$\omega(G)$ = clique number = $\max\{\#$clique$\}$



$$\omega(C_5) = \alpha(C_5) = 2.$$

$\forall G$ with 6 vertices, $\omega(G) \geq 3$ or $\alpha(G) \geq 3$.

These fact can be conveniently described by the Ramsey number:

$$R(3,3) = 6.$$

The smallest number of vertices required to guarantee $\alpha \geq 3$ or $\omega \geq 3$ (precise definition in the next slide).

Ramsey Numbers and a Generalization

Definition

The Ramsey number $R(m_1, m_2)$ is defined as:

$$R(m_1, m_2)$$
$$= \min\{n \mid |V(G)| = n \implies \omega(G) \geq m_1 \text{ or } \alpha(G) \geq m_2\}$$
$$= \min\{n \mid |V(G)| = n \implies \omega(G) \geq m_1 \text{ or } \omega(\bar{G}) \geq m_2\}$$

A graph with n vertices defines a partition of $E(K_n)$ into 2 parts, “edges” and “non-edges”.

Generalized Ramsey numbers $R(m_1, m_2, \ldots, m_k)$ can be defined if we consider partitions of $E(K_n)$ into k parts, i.e., (not necessarily proper) edge-colorings.

Definition (Complementary Ramsey numbers)

We write by $[n] = \{1, 2, \dots, n\}$, and denote by $E(K_n) = \binom{[n]}{2}$ the set of 2-subsets of $[n]$. The set of k-edge-coloring of K_n is denoted by $C(n, k)$:

$$C(n, k) = \{f \mid f : E(K_n) \to [k]\}.$$

We abbreviate

$$\omega_i(f) = \omega([n], f^{-1}(i)), \quad \alpha_i(f) = \alpha([n], f^{-1}(i)).$$

$$R(m_1, \dots, m_k) = \min\{n \mid \forall f \in C(n, k),\ \exists i \in [k],\ \omega_i(f) \ge m_i\}$$
$$\bar{R}(m_1, \dots, m_k) = \min\{n \mid \forall f \in C(n, k),\ \exists i \in [k],\ \alpha_i(f) \ge m_i\}$$

The last one is called the complemtary Ramsey number.

$$\bar{R}(m_1, m_2) = R(m_2, m_1) = R(m_1, m_2).$$

Geometric Application

Given a metric space (X, d) and a positive integer k, classify subsets Y of X with the largest size subject to

$$|\{d(x,y) \mid x,y \in Y,\ x \neq y\}| \leq k.$$

For example, $X = \mathbb{R}^n$, $k = 1 \implies$ regular simplex.
The method is by induction on k.

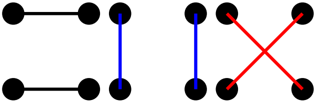
The distance function d defines a k-edge-coloring of the complete graph on Y.

If

$$\bar{R}(\underbrace{m, m, \ldots, m}_{k}) \leq |Y|,$$

then Y must contain an m-subset having only $(k-1)$ distances (so we can expect to use already obtained results for $k-1$).

$\bar{R}(3,3,3) = 5$ by factorization



- K_4 has a 3-edge-coloring f into $2K_2$ (a 1-factorization). Then $\alpha_i(f) = 2$ for $i = 1, 2, 3$. This implies
$$\bar{R}(3,3,3) > 4.$$
- If f is a 3-edge-coloring of K_5, then some color i has at most 3 edges, so $\alpha_i(f) \geq 3$.

The argument can be generalized to give:

Theorem

If K_{mn} is factorable into k copies of nK_m, then $\bar{R}(\underbrace{n+1, \dots, n+1}_{k}) = mn + 1$.

Setting $m = n = 2$ and $k = 3$, we obtain $\bar{R}(3,3,3) = 5$.

Factorizations

Theorem

If K_{mn} is factorable into k copies of nK_m, then $\bar{R}(\underbrace{n+1,\ldots,n+1}_{k}) = mn+1$.

- Setting $m=2$, $k=2n-1$, the existence of a 1-factorization in K_{2n} implies $\bar{R}(\underbrace{n+1,\ldots,n+1}_{2n-1}) = 2n+1$.
- Setting $m=3$, $n=2t+1$, $k=3t+1$, the existence of a Kirkman triple system in K_{3n} implies $\bar{R}(\underbrace{2t+2,\ldots,2t+2}_{3t+1}) = 6t+4$.
- Setting $m=n$, $k=n+1$, if $n-1$ MOLS of order n exist, then $\bar{R}(\underbrace{n+1,\ldots,n+1}_{n+1}) = n^2+1$.

Theorem

There exist $n-1$ MOLS of order n (K_{n^2} into $n+1$ nK_n's) iff $\bar{R}(\underbrace{n+1,\ldots,n+1}_{n+1}) = n^2+1$.

Non-uniform case, thanks to Turán graphs:

Theorem

Let k and $N>1$ be integers. Suppose that K_N is factorable into $H_1, H_2, \ldots, H_k$ where

$$H_i \cong r_i K_{q_i+1} \cup (n_i - r_i) K_{q_i},$$
$$N = n_i q_i + r_i,$$
$$0 \le r_i < n_i$$

Assume further that $(n_i - r_i - 1) q_i > 0$ for some $i \in [k]$. Then

$$\bar{R}(n_1+1, n_2+1, \ldots, n_k+1) = N+1$$

Table of small complementary Ramsey numbers

k	3	4	5	6	7	8
$\bar{R}(k,3,3)$	5	5	5	6	$\cdots$	$\cdots$
$\bar{R}(k,4,3)$	5	7	8	8	9	$\cdots$

We abbreviate

$$\bar{R}(m;k) = \bar{R}(\underbrace{m,\ldots,m}_{k}).$$

k	3	4	5	6	7	8	9	10	11 $\cdots$ 15	16
$\bar{R}(3;k)$	5	3	$\cdots$							
$\bar{R}(4;k)$	10	10	7	5	4	$\cdots$				
$\bar{R}(5;k)$	?	?	17	10	9	6	6	6	5 $\cdots$	
$\bar{R}(6;k)$	?	?	?	26	16	11	11	8	7 $\cdots$ 7	6