Grassmann Graphs

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November 21, 2013

The formula for the binomial coefficient $\binom{n}{i}$ is found by counting

$$\#\{(Y, y_1, \dots, y_i) \mid Y = \{y_1, \dots, y_i\} \subset V,\ \#Y = i\}$$

in two ways, where $\#V = n$. Indeed,

$$\#\{Y \mid Y \subset V,\ \#Y = i\} \cdot i(i-1)\cdots 1 = n(n-1)\cdots(n-i+1),$$

and hence $\binom{n}{i} = \frac{n(n-1)\cdots(n-i+1)}{i!}$.

Notation.

$$\begin{aligned}
\begin{bmatrix} W \\ i \end{bmatrix} &: \text{the set of all } i\text{-dimensional subspaces of a vector space } W \text{ over } \mathbb{F}_q \\
\left\langle \begin{matrix} m, j \\ i \end{matrix} \right\rangle &= (q^m - q^j)(q^m - q^{j+1})\cdots(q^m - q^{j+i-1}) \\
&= q^{ij} \left\langle \begin{matrix} m-j, 0 \\ i \end{matrix} \right\rangle \qquad (1) \\
\begin{bmatrix} m \\ i \end{bmatrix} &= \left\langle \begin{matrix} m, 0 \\ i \end{matrix} \right\rangle \left\langle \begin{matrix} i, 0 \\ i \end{matrix} \right\rangle^{-1} \\
\langle y_1, \dots, y_i \rangle &: \text{the linear span of } y_1, \dots, y_i \in V
\end{aligned}$$

Observe

$$\left\langle \begin{matrix} m, 0 \\ i \end{matrix} \right\rangle \left\langle \begin{matrix} m, i \\ j \end{matrix} \right\rangle = \left\langle \begin{matrix} m, 0 \\ i+j \end{matrix} \right\rangle, \qquad (2)$$

and

$$\begin{aligned}
\begin{bmatrix} m \\ m-i \end{bmatrix} &= \frac{\left\langle \begin{smallmatrix} m,0 \\ m-i \end{smallmatrix} \right\rangle \left\langle \begin{smallmatrix} m,m-i \\ i \end{smallmatrix} \right\rangle}{\left\langle \begin{smallmatrix} m-i,0 \\ m-i \end{smallmatrix} \right\rangle \left\langle \begin{smallmatrix} m,m-i \\ i \end{smallmatrix} \right\rangle} \\
&= \frac{\left\langle \begin{smallmatrix} m,0 \\ m \end{smallmatrix} \right\rangle}{\left\langle \begin{smallmatrix} m-i,0 \\ m-i \end{smallmatrix} \right\rangle \left\langle \begin{smallmatrix} m,m-i \\ i \end{smallmatrix} \right\rangle} && \text{(by (2))} \\
&= \frac{\left\langle \begin{smallmatrix} m,0 \\ m \end{smallmatrix} \right\rangle}{q^{-i(m-i)} \left\langle \begin{smallmatrix} m,i \\ m-i \end{smallmatrix} \right\rangle q^{i(m-i)} \left\langle \begin{smallmatrix} i,0 \\ i \end{smallmatrix} \right\rangle} && \text{(by (1))} \\
&= \frac{\left\langle \begin{smallmatrix} m,0 \\ i \end{smallmatrix} \right\rangle \left\langle \begin{smallmatrix} m,i \\ m-i \end{smallmatrix} \right\rangle}{\left\langle \begin{smallmatrix} m,i \\ m-i \end{smallmatrix} \right\rangle \left\langle \begin{smallmatrix} i,0 \\ i \end{smallmatrix} \right\rangle} && \text{(by (2))} \\
&= \begin{bmatrix} m \\ i \end{bmatrix}. && (3)
\end{aligned}$$

Let V be an n-dimensional vector space over $\mathbb{F}_q$, and fix $X \in \left[\begin{smallmatrix} V \\ j \end{smallmatrix}\right]$. If Z is a subspace of V, then

$$\#\{y \in V \mid \dim\langle Z, y \rangle = \dim Z + 1\} = \#\{y \in V \mid y \notin Z\}$$

$$= \#V - \#Z$$
$$= q^n - q^{\dim Z}. \tag{4}$$

Thus, if $\dim Z = k$, then

$$\#\{(y_{k+1}, \ldots, y_i) \mid \dim\langle Z, y_{k+1}, \ldots, y_i\rangle = i\} = \left\langle \begin{matrix} n, k \\ i-k \end{matrix} \right\rangle. \tag{5}$$

One can regard $\left\langle \begin{smallmatrix} n,k \\ i-k \end{smallmatrix} \right\rangle$ as the number of way of adding $i-k$ vectors to a k-dimensional subspace to form an i-dimensional subspace in an n-dimensional space. Note that, in the set-theoretical setting, the number of ways of adding $i-k$ elements to a k-element subset to form an i-element subset in an n-element set is

$$(n-k)(n-(k+1))\cdots(n-(i-1))$$

which looks similar to

$$\left\langle \begin{matrix} n, k \\ i-k \end{matrix} \right\rangle = (q^n - q^k)(q^n - q^{k+1})\cdots(q^n - q^{i-1}).$$

A proof of (5) can be given using induction on $i-k$. Indeed, (5) is the same as (4) if $i = k+1$. Suppose $i - k > 1$. Then

$$\begin{aligned}
&\#\{(y_{k+1}, \ldots, y_i) \mid \dim\langle Z, y_{k+1}, \ldots, y_i\rangle = i\} \\
&= \sum_{\dim\langle Z, y_{k+1}, \ldots, y_{i-1}\rangle = i-1} \#\{y_i \mid \dim\langle Z, y_{k+1}, \ldots, y_i\rangle = i\} \\
&= \sum_{\dim\langle Z, y_{k+1}, \ldots, y_{i-1}\rangle = i-1} (q^n - q^{i-1}) && \text{(by (4))} \\
&= \left\langle \begin{matrix} n, i-1 \\ 1 \end{matrix} \right\rangle \#\{(y_{k+1}, \ldots, y_{i-1}) \mid \dim\langle Z, y_{k+1}, \ldots, y_{i-1}\rangle = i-1\} \\
&= \left\langle \begin{matrix} n, i-1 \\ 1 \end{matrix} \right\rangle \left\langle \begin{matrix} n, k \\ i-1-k \end{matrix} \right\rangle && \text{(by induction)} \\
&= \left\langle \begin{matrix} n, k \\ i-k \end{matrix} \right\rangle.
\end{aligned}$$

1. Count $\{(Y, y_1, \ldots, y_i) \mid Y = \langle y_1, \ldots, y_i\rangle \in \left[\begin{smallmatrix} V \\ i \end{smallmatrix}\right]\}$ to derive $\#\left[\begin{smallmatrix} V \\ i \end{smallmatrix}\right] = \left[\begin{smallmatrix} n \\ i \end{smallmatrix}\right]$.

$$\begin{aligned}
&\#\{(Y, y_1, \ldots, y_i) \mid Y = \langle y_1, \ldots, y_i\rangle \in \begin{bmatrix} V \\ i \end{bmatrix}\} \\
&= \#\{(y_1, \ldots, y_i) \mid \langle y_1, \ldots, y_i\rangle \in \begin{bmatrix} V \\ i \end{bmatrix}\} \\
&= \left\langle \begin{matrix} n, 0 \\ i \end{matrix} \right\rangle && \text{(by (5))},
\end{aligned}$$

also,

$$= \sum_{Y \in \left[\begin{smallmatrix} V \\ i \end{smallmatrix}\right]} \#\{(y_1, \ldots, y_i) \mid Y = \langle y_1, \ldots, y_i\rangle\}$$

$$
\begin{aligned}
&= \sum_{Y \in \left[\begin{smallmatrix} V \\ i \end{smallmatrix}\right]} \#\{(y_1, \dots, y_i) \mid y_1, \dots, y_i \in Y,\ \dim\langle y_1, \dots, y_i \rangle = i\} \\
&= \sum_{Y \in \left[\begin{smallmatrix} V \\ i \end{smallmatrix}\right]} \left\langle \begin{matrix} i, 0 \\ i \end{matrix} \right\rangle && \text{(by (5))} \\
&= \left\langle \begin{matrix} i, 0 \\ i \end{matrix} \right\rangle \cdot \# \begin{bmatrix} V \\ i \end{bmatrix}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\# \begin{bmatrix} V \\ i \end{bmatrix} &= \frac{\left\langle \begin{smallmatrix} n, 0 \\ i \end{smallmatrix} \right\rangle}{\left\langle \begin{smallmatrix} i, 0 \\ i \end{smallmatrix} \right\rangle} \\
&= \begin{bmatrix} n \\ i \end{bmatrix}. \qquad (6)
\end{aligned}
$$

2. Count $\{(Y, y_1, \dots, y_i) \mid Y = \langle y_1, \dots, y_i \rangle \in \left[\begin{smallmatrix} V \\ i \end{smallmatrix}\right],\ X \cap Y = 0\}$ to derive $\#\{Y \in \left[\begin{smallmatrix} V \\ i \end{smallmatrix}\right] \mid X \cap Y = 0\} = q^{ij} \left[\begin{smallmatrix} n-j \\ i \end{smallmatrix}\right]$.

$$
\begin{aligned}
&\#\{(Y, y_1, \dots, y_i) \mid Y = \langle y_1, \dots, y_i \rangle \in \begin{bmatrix} V \\ i \end{bmatrix},\ X \cap Y = 0\} \\
&= \#\{(y_1, \dots, y_i) \mid \langle y_1, \dots, y_i \rangle \in \begin{bmatrix} V \\ i \end{bmatrix},\ X \cap \langle y_1, \dots, y_i \rangle = 0\} \\
&= \#\{(y_1, \dots, y_i) \mid \dim\langle X, y_1, \dots, y_i \rangle = j + i\} \\
&= \left\langle \begin{matrix} n, j \\ i \end{matrix} \right\rangle && \text{(by (5)),}
\end{aligned}
$$

also,

$$
\begin{aligned}
&= \sum_{\substack{Y \in \left[\begin{smallmatrix} V \\ i \end{smallmatrix}\right] \\ X \cap Y = 0}} \#\{(y_1, \dots, y_i) \mid Y = \langle y_1, \dots, y_i \rangle\} \\
&= \sum_{\substack{Y \in \left[\begin{smallmatrix} V \\ i \end{smallmatrix}\right] \\ X \cap Y = 0}} \#\{(y_1, \dots, y_i) \mid y_1, \dots, y_i \in Y,\ \dim\langle y_1, \dots, y_i = i \rangle\} \\
&= \sum_{\substack{Y \in \left[\begin{smallmatrix} V \\ i \end{smallmatrix}\right] \\ X \cap Y = 0}} \left\langle \begin{matrix} i, 0 \\ i \end{matrix} \right\rangle && \text{(by (5))} \\
&= \left\langle \begin{matrix} i, 0 \\ i \end{matrix} \right\rangle \cdot \#\{Y \in \begin{bmatrix} V \\ i \end{bmatrix} \mid X \cap Y = 0\}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\#\{Y \in \begin{bmatrix} V \\ i \end{bmatrix} \mid X \cap Y = 0\} &= \frac{\left\langle \begin{smallmatrix} n, j \\ i \end{smallmatrix} \right\rangle}{\left\langle \begin{smallmatrix} i, 0 \\ i \end{smallmatrix} \right\rangle} \\
&= \frac{q^{ij} \left\langle \begin{smallmatrix} n-j, 0 \\ i \end{smallmatrix} \right\rangle}{\left\langle \begin{smallmatrix} i, 0 \\ i \end{smallmatrix} \right\rangle} && \text{(by (1))}
\end{aligned}
$$

$$= q^{ij}\begin{bmatrix} n-j \\ i \end{bmatrix}. \tag{7}$$

3. Count $\{(Y, y_{j+1}, \ldots, y_i) \mid Y = \langle X, y_{j+1}, \ldots, y_i\rangle \in \left[\begin{smallmatrix} V \\ i \end{smallmatrix}\right]\}$ to derive $\#\{Y \in \left[\begin{smallmatrix} V \\ i \end{smallmatrix}\right] \mid X \subset Y\} = \left[\begin{smallmatrix} n-j \\ i-j \end{smallmatrix}\right]$.

$$\begin{aligned}
&\#\{(Y, y_{j+1}, \ldots, y_i) \mid Y = \langle X, y_{j+1}, \ldots, y_i\rangle \in \begin{bmatrix} V \\ i \end{bmatrix}\} \\
&= \#\{(y_{j+1}, \ldots, y_i) \mid \dim\langle X, y_{j+1}, \ldots, y_i\rangle = i\} \\
&= \left\langle \begin{matrix} n, j \\ i-j \end{matrix} \right\rangle \qquad \text{(by (5))},
\end{aligned}$$

also,

$$\begin{aligned}
&= \sum_{\substack{Y \in \left[\begin{smallmatrix} V \\ i \end{smallmatrix}\right] \\ X \subset Y}} \#\{(y_{j+1}, \ldots, y_i) \mid Y = \langle X, y_{j+1}, \ldots, y_i\rangle\} \\
&= \sum_{\substack{Y \in \left[\begin{smallmatrix} V \\ i \end{smallmatrix}\right] \\ X \subset Y}} \#\{(y_{j+1}, \ldots, y_i) \mid y_{j+1}, \ldots, y_i \in Y,\ \dim\langle X, y_{j+1}, \ldots, y_i\rangle = i\} \\
&= \sum_{\substack{Y \in \left[\begin{smallmatrix} V \\ i \end{smallmatrix}\right] \\ X \subset Y}} \left\langle \begin{matrix} i, j \\ i-j \end{matrix} \right\rangle \qquad \text{(by (5))} \\
&= \left\langle \begin{matrix} i, j \\ i-j \end{matrix} \right\rangle \cdot \#\{Y \in \begin{bmatrix} V \\ i \end{bmatrix} \mid X \subset Y\}.
\end{aligned}$$

Thus

$$\begin{aligned}
\#\{Y \in \begin{bmatrix} V \\ i \end{bmatrix} \mid X \subset Y\} &= \frac{\left\langle \begin{smallmatrix} n, j \\ i-j \end{smallmatrix} \right\rangle}{\left\langle \begin{smallmatrix} i, j \\ i-j \end{smallmatrix} \right\rangle} \\
&= \frac{q^{j(i-j)}\left\langle \begin{smallmatrix} n-j, 0 \\ i \end{smallmatrix} \right\rangle}{q^{j(i-j)}\left\langle \begin{smallmatrix} i-j, 0 \\ i-j \end{smallmatrix} \right\rangle} \qquad \text{(by (1))} \\
&= \begin{bmatrix} n-j \\ i-j \end{bmatrix}.
\end{aligned} \tag{8}$$

4. Count $\{(Y, W, y_{m+1}, \ldots, y_i) \mid Y = \langle W, y_{m+1}, \ldots, y_i\rangle \in \left[\begin{smallmatrix} V \\ i \end{smallmatrix}\right],\ X \cap Y = W \in \left[\begin{smallmatrix} X \\ m \end{smallmatrix}\right]\}$ to derive $\#\{Y \in \left[\begin{smallmatrix} V \\ i \end{smallmatrix}\right] \mid \dim X \cap Y = m\} = q^{(i-m)(j-m)}\left[\begin{smallmatrix} j \\ m \end{smallmatrix}\right]\left[\begin{smallmatrix} n-j \\ i-m \end{smallmatrix}\right]$.

$$\begin{aligned}
&\#\{(Y, W, y_{m+1}, \ldots, y_i) \mid Y = \langle W, y_{m+1}, \ldots, y_i\rangle \in \begin{bmatrix} V \\ i \end{bmatrix},\ X \cap Y = W \in \begin{bmatrix} X \\ m \end{bmatrix}\} \\
&= \sum_{W \in \left[\begin{smallmatrix} X \\ m \end{smallmatrix}\right]} \{(Y, y_{m+1}, \ldots, y_i) \mid Y = \langle W, y_{m+1}, \ldots, y_i\rangle \in \begin{bmatrix} V \\ i \end{bmatrix},\ X \cap Y = W\}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{W \in \left[{X \atop m}\right]} \{(y_{m+1}, \ldots, y_i) \mid \langle W, y_{m+1}, \ldots, y_i \rangle \in \begin{bmatrix} V \\ i \end{bmatrix}, \\
&\quad X \cap \langle W, y_{m+1}, \ldots, y_i \rangle = W\} \\
&= \sum_{W \in \left[{X \atop m}\right]} \{(y_{m+1}, \ldots, y_i) \mid \dim\langle W, y_{m+1}, \ldots, y_i \rangle = i, \\
&\quad \dim X \cap \langle W, y_{m+1}, \ldots, y_i \rangle = m\} \\
&= \sum_{W \in \left[{X \atop m}\right]} \{(y_{m+1}, \ldots, y_i) \mid \dim\langle W, y_{m+1}, \ldots, y_i \rangle = i, \\
&\quad j + \dim\langle W, y_{m+1}, \ldots, y_i \rangle - \dim\langle X, W, y_{m+1}, \ldots, y_i \rangle = m\} \\
&= \sum_{W \in \left[{X \atop m}\right]} \{(y_{m+1}, \ldots, y_i) \mid \dim\langle W, y_{m+1}, \ldots, y_i \rangle = i, \\
&\quad \dim\langle X, y_{m+1}, \ldots, y_i \rangle = j + i - m\} \\
&= \sum_{W \in \left[{X \atop m}\right]} \{(y_{m+1}, \ldots, y_i) \mid \dim\langle X, y_{m+1}, \ldots, y_i \rangle = j + i - m\} \\
&= \sum_{W \in \left[{X \atop m}\right]} \left\langle {n, j \atop i - m} \right\rangle && \text{(by (5))} \\
&= \begin{bmatrix} j \\ m \end{bmatrix} \left\langle {n, j \atop i - m} \right\rangle && \text{(by (6))},
\end{aligned}
$$

also,

$$
\begin{aligned}
&= \sum_{\substack{Y \in \left[{V \atop i}\right] \\ X \cap Y \in \left[{X \atop m}\right]}} \#\{(W, y_{m+1}, \ldots, y_i) \mid Y = \langle W, y_{m+1}, \ldots, y_i \rangle,\ X \cap Y = W\} \\
&= \sum_{\substack{Y \in \left[{V \atop i}\right] \\ X \cap Y \in \left[{X \atop m}\right]}} \#\{(y_{m+1}, \ldots, y_i) \mid Y = \langle X \cap Y, y_{m+1}, \ldots, y_i \rangle\} \\
&= \sum_{\substack{Y \in \left[{V \atop i}\right] \\ X \cap Y \in \left[{X \atop m}\right]}} \#\{(y_{m+1}, \ldots, y_i) \mid y_{m+1}, \ldots, y_i \in Y, \\
&\quad \dim\langle X \cap Y, y_{m+1}, \ldots, y_i \rangle = i\} \\
&= \sum_{\substack{Y \in \left[{V \atop i}\right] \\ X \cap Y \in \left[{X \atop m}\right]}} \left\langle {i, m \atop i - m} \right\rangle \\
&= \left\langle {i, m \atop i - m} \right\rangle \cdot \#\{Y \in \begin{bmatrix} V \\ i \end{bmatrix} \mid \dim X \cap Y = m\}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\#\{Y \in \begin{bmatrix} V \\ i \end{bmatrix} \mid \dim X \cap Y = m\} &= \begin{bmatrix} j \\ m \end{bmatrix} \frac{\left\langle {n,j \atop i-m} \right\rangle}{\left\langle {i,m \atop i-m} \right\rangle} \\
&= \begin{bmatrix} j \\ m \end{bmatrix} \frac{q^{j(i-m)} \left\langle {n-j,0 \atop i-m} \right\rangle}{q^{m(i-m)} \left\langle {i-m,0 \atop i-m} \right\rangle} \qquad \text{(by (1))} \\
&= q^{(i-m)(j-m)} \begin{bmatrix} j \\ m \end{bmatrix} \begin{bmatrix} n-j \\ i-m \end{bmatrix}. \qquad (9)
\end{aligned}
$$

Definition. Let $n \geq 2d$. The **Grassmann graph** $J_q(n, d)$ is the graph with vertex set $\begin{bmatrix} V \\ d \end{bmatrix}$, where V is a vector space of dimension n over $\mathbb{F}_q$, and two vertices X, Y are adjacent whenever $\dim X \cap Y = d - 1$.

Let $X, Y, Z \in \begin{bmatrix} V \\ d \end{bmatrix}$ and assume $\dim Y \cap Z = d - 1$. Then

$$
\begin{aligned}
\dim X \cap Y &\geq \dim X \cap Y \cap Z \\
&= \dim(X \cap Z) \cap (Y \cap Z) \\
&= \dim X \cap Z + \dim Y \cap Z - \dim(X \cap Z + Y \cap Z) \\
&\geq \dim X \cap Z + \dim Y \cap Z - \dim Z \\
&= \dim X \cap Z - 1. \qquad (10)
\end{aligned}
$$

Thus

$$\dim X \cap Y = \dim X \cap Z - 1 \implies X \cap Y \subset Z = X \cap Z + Y \cap Z. \qquad (11)$$

For $i = 0, 1, \dots, d$, we claim

$$(\text{the distance between } X \text{ and } Y) = d - \dim X \cap Y.$$

Indeed, if $\dim X \cap Y = d - i$, then take a basis $x_1, \dots, x_i, u_1, \dots, u_{d-i}, y_1, \dots, y_i$ of $X + Y$ in such a way that

$$
\begin{aligned}
X &= \langle x_1, \dots, x_i, u_1, \dots, u_{d-i} \rangle, \\
Y &= \langle u_1, \dots, u_{d-i}, y_1, \dots, y_i \rangle
\end{aligned}
$$

Then

$$
\begin{aligned}
X_0 &= X, \\
X_1 &= \langle x_2, \dots, x_i, u_1, \dots, u_{d-i}, y_1 \rangle, \\
&\vdots \\
X_{i-1} &= \langle x_i, u_1, \dots, u_{d-i}, y_1, \dots, y_{i-1} \rangle, \\
X_i &= Y
\end{aligned}
$$

is a path of length i from X to Y. Thus the distance between X and Y is at most i. This means

$$(\text{the distance between } X \text{ and } Y) \leq d - \dim X \cap Y.$$

The reverse inequality

$$(\text{the distance between } X \text{ and } Y) \geq d - \dim X \cap Y \tag{12}$$

can be proved by induction on the distance. Indeed, it is trivially true for distance 0 or 1. Suppose X and Y are at distance $i \geq 2$. Choose Z at distance $i-1$ from X, and distance 1 from Y. Then by induction, $i - 1 \geq d - \dim X \cap Z$. The inequality (10) then implies $\dim X \cap Y \geq \dim X \cap Z - 1 \geq d - i$, hence (12) holds.

Define

$$R_i = \{(X, Y) \mid X, Y \in \begin{bmatrix} V \\ d \end{bmatrix},\ \dim X \cap Y = d - i\}.$$

5. $GL(V)$ acts transitively on R_i for each $i = 0, 1, \dots, d$. Indeed, let $(X, Y), (X', Y') \in R_i$. Take bases

$$\begin{aligned} &x_1, \dots, x_i, u_1, \dots, u_{d-i}, y_1, \dots, y_i, v_{d+i+1}, \dots, v_n, \text{ and} \\ &x'_1, \dots, x'_i, u'_1, \dots, u'_{d-i}, y'_1, \dots, y'_i, v'_{d+i+1}, \dots, v'_n \end{aligned}$$

of V in such a way that

$$\begin{aligned} X &= \langle x_1, \dots, x_i, u_1, \dots, u_{d-i} \rangle, \\ Y &= \langle u_1, \dots, u_{d-i}, y_1, \dots, y_i \rangle, \\ X' &= \langle x'_1, \dots, x'_i, u'_1, \dots, u'_{d-i} \rangle, \\ Y' &= \langle u'_1, \dots, u'_{d-i}, y'_1, \dots, y'_i \rangle \end{aligned}$$

Then the invertible linear transformation $g \in GL(V)$ defined by

$$x_j \mapsto x'_j, \quad u_j \mapsto u'_j, \quad y_j \mapsto y'_j, \quad v_j \mapsto v'_j$$

maps X to X' and Y to Y'.

6. Let $k_i = \#\{Y \mid (X, Y) \in R_i\}$. Then

$$k_i = q^{i^2} \begin{bmatrix} d \\ i \end{bmatrix} \begin{bmatrix} n-d \\ i \end{bmatrix}. \tag{13}$$

This is a special case of (9) by setting (i, m, j) to be $(d, d-i, d)$.

Suppose $(X, Y) \in R_i$, $(X, Z) \in R_{i-1}$, $(Z, Y) \in R_1$. Then there is a bijection from $\{Z \mid (X, Z) \in R_{i-1},\ (Z, Y) \in R_1\}$ to the set

$$\{(X', Y') \mid X \cap Y \subset X' \in \begin{bmatrix} X \\ d-i+1 \end{bmatrix},\ X \cap Y \subset Y' \in \begin{bmatrix} Y \\ d-1 \end{bmatrix}\}$$

defined by

$$\phi : Z \mapsto (X \cap Z, Y \cap Z), \quad \psi : (X', Y') \mapsto X' + Y'.$$

Indeed, $\psi \circ \phi = \text{id}$ since equality holds in (10). To show $\phi \circ \psi = \text{id}$, we need $X \cap (X' + Y') = X'$ and $Y \cap (X' + Y') = Y'$ whenever $X \cap Y \subset X'$ and $X \cap Y \subset Y'$. If $x = x' + y' \in X \cap (X' + Y')$, then $y' \in (X + X') \cap Y' = X \cap Y' \subset X \cap Y \subset X'$, so $x = x' + y' \in X'$. This proves

$X \cap (X' + Y') \subset X'$, while the reverse containment is obvious. Similarly, one obtains $Y \cap (X' + Y') = Y'$.

Let

$$c_i = \#\{Z \mid (X, Z) \in R_{i-1},\ (Z, Y) \in R_1\},$$

where $(X, Y) \in R_i$. Then

$$\begin{aligned} c_i &= \#\{(X', Y') \mid X \cap Y \subset X' \in \begin{bmatrix} X \\ d-i+1 \end{bmatrix},\ X \cap Y \subset Y' \in \begin{bmatrix} Y \\ d-1 \end{bmatrix}\} \\ &= \#\{X' \in \begin{bmatrix} X \\ d-i+1 \end{bmatrix} \mid X \cap Y \subset X'\} \times \#\{Y' \in \begin{bmatrix} Y \\ d-1 \end{bmatrix} \mid X \cap Y \subset Y'\} \\ &= \begin{bmatrix} i \\ 1 \end{bmatrix} \begin{bmatrix} i \\ i-1 \end{bmatrix} && \text{(by (8))} \\ &= \begin{bmatrix} i \\ 1 \end{bmatrix}^2 && \text{(by (3))}. \quad (14) \end{aligned}$$

Let

$$b_i = \#\{Z \mid (X, Z) \in R_{i+1},\ (Z, Y) \in R_1\},$$

where $(X, Y) \in R_i$. Note that this number is independent of the choice of $(X, Y) \in R_i$ since $GL(V)$ acts transitively on R_i. Since, for a fixed $X \in \begin{bmatrix} V \\ d \end{bmatrix}$,

$$\begin{aligned} &\#\{(Y, Z) \in R_1 \mid (X, Y) \in R_i,\ (X, Z) \in R_{i+1}\} \\ &= \sum_{\substack{Z \in \begin{bmatrix} V \\ d \end{bmatrix} \\ (X,Z) \in R_{i+1}}} \#\{Y \mid (Y, Z) \in R_1,\ (X, Y) \in R_i\} \\ &= \sum_{\substack{Z \in \begin{bmatrix} V \\ d \end{bmatrix} \\ (X,Z) \in R_{i+1}}} c_{i+1} \\ &= k_{i+1} c_{i+1}, \end{aligned}$$

also,

$$\begin{aligned} &= \sum_{\substack{Y \in \begin{bmatrix} V \\ d \end{bmatrix} \\ (X,Y) \in R_i}} \#\{Z \mid (Y, Z) \in R_1,\ (X, Z) \in R_{i+1}\} \\ &= \sum_{\substack{Y \in \begin{bmatrix} V \\ d \end{bmatrix} \\ (X,Y) \in R_i}} b_i \\ &= k_i b_i, \end{aligned}$$

we have

$$\begin{aligned} b_i &= \frac{c_{i+1} k_{i+1}}{k_i} \\ &= \frac{\begin{bmatrix} i+1 \\ 1 \end{bmatrix}^2 q^{(i+1)^2} \begin{bmatrix} d \\ i+1 \end{bmatrix} \begin{bmatrix} n-d \\ i+1 \end{bmatrix}}{q^{i^2} \begin{bmatrix} d \\ i \end{bmatrix} \begin{bmatrix} n-d \\ i \end{bmatrix}} && \text{(by (13), (14))} \end{aligned}$$

$$
\begin{aligned}
&= q^{2i+1} \frac{\left\langle {i+1,0 \atop 1} \right\rangle^2 \left\langle {d,0 \atop i+1} \right\rangle \left\langle {n-d,0 \atop i+1} \right\rangle \left\langle {i,0 \atop i} \right\rangle^2}{\left\langle {1,0 \atop 1} \right\rangle^2 \left\langle {i+1,0 \atop i+1} \right\rangle^2 \left\langle {d,0 \atop i} \right\rangle \left\langle {n-d,0 \atop i} \right\rangle} \\
&= q^{2i+1} \left(\frac{\left\langle {i+1,0 \atop 1} \right\rangle \left\langle {i,0 \atop i} \right\rangle}{\left\langle {1,0 \atop 1} \right\rangle \left\langle {i+1,0 \atop i+1} \right\rangle} \right)^2 \frac{\left\langle {d,0 \atop i+1} \right\rangle \left\langle {n-d,0 \atop i+1} \right\rangle}{\left\langle {d,0 \atop i} \right\rangle \left\langle {n-d,0 \atop i} \right\rangle} \\
&= q^{2i+1} \left(\frac{\left\langle {i,0 \atop i} \right\rangle}{\left\langle {1,0 \atop 1} \right\rangle \left\langle {i+1,1 \atop i} \right\rangle} \right)^2 \left\langle {d,i \atop 1} \right\rangle \left\langle {n-d,i \atop 1} \right\rangle && \text{(by (2))} \\
&= q^{2i+1} \left(\frac{1}{\left\langle {1,0 \atop 1} \right\rangle q^i} \right)^2 q^{2i} \left\langle {d-i,0 \atop 1} \right\rangle \left\langle {n-d-i,0 \atop 1} \right\rangle && \text{(by (1))} \\
&= q^{2i+1} \frac{\left\langle {d-i,0 \atop 1} \right\rangle \left\langle {n-d-i,0 \atop 1} \right\rangle}{\left\langle {1,0 \atop 1} \right\rangle \left\langle {1,0 \atop 1} \right\rangle} \\
&= q^{2i+1} \begin{bmatrix} d-i \\ 1 \end{bmatrix} \begin{bmatrix} n-d-i \\ 1 \end{bmatrix}.
\end{aligned}
$$