Twisted symplectic polar graphs

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(joint work with Frédéric Vanhove)

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In Takuya Ikuta's talk (Nov. 18)

$$P = \begin{array}{c} \\ \mathbb{R}\mathbf{1} \\ V_1 \\ V_2 \\ V_3 \end{array} \begin{array}{c} \begin{array}{cccc} I & A_1 & A_2 & A_3 \end{array} \\ \begin{pmatrix} 1 & \frac{q^2}{2} - q & \frac{q^2}{2} & q-2 \\ 1 & \frac{q}{2} & -\frac{q}{2} & -1 \\ 1 & -\frac{q}{2}+1 & -\frac{q}{2} & q-2 \\ 1 & -\frac{q}{2} & \frac{q}{2} & -1 \end{pmatrix} \end{array}$$

$$\mathbb{R}^{q^2-1} = \mathbb{R}\mathbf{1} \oplus V_1 \oplus V_2 \oplus V_3$$

$$J = I + A_1 + A_2 + A_3$$

$$H = I + \alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3$$

: complex Hadamard

Eiichi Bannai: are there many a.s. with this eigenmatrix?

E. van Dam (1999): Table of 3-class association schemes
$\rightarrow$ a.s. with eigenmatrix P with $q = 4, 8$ have complex Hadamard
$\rightarrow$ a.s. with eigenmatrix P with any $q \geq 4$ (power of 2) have complex Hadamard

An example is given in
Brouwer–Cohen–Neumaier,
“Distance-Regular Graphs”, Sect. 12.1.1

Q: non degenerate quadratic form on $V = \mathbb{F}_q^3$
$\to$ quadric $\mathcal{Q} = \{\langle x\rangle \mid Q(x) = 0\}$ on $PG(2,q)$
with nucleus $\langle \nu\rangle = V^\perp$.

$$X = \{\langle x\rangle \in PG(2,q) \mid \langle x\rangle \notin \mathcal{Q},\ \langle x\rangle \neq \langle \nu\rangle\}$$

Then $|X| = q^2 - 1$.

$$R_1 = \{(\langle x\rangle, \langle y\rangle) \mid \langle x, y\rangle : \text{ secant}\}$$
$$R_2 = \{(\langle x\rangle, \langle y\rangle) \mid \langle x, y\rangle : \text{ exterior}\}$$
$$R_3 = \{(\langle x\rangle, \langle y\rangle) \mid \langle x, y\rangle : \text{ tangent}\}$$

BCN gives the computation of intersection numbers (sketch), remarks "the proof has to be a geometric one." $\to$ painful

Frédéric Vanhove (U. Ghent) visited me (Sep. 6–Oct. 4).

$$O(3,q) \cong Sp(2,q) \quad (q : \text{even})$$

$$P = \begin{array}{c} \\ \mathbb{R}\mathbf{1} \\ V_1 \\ V_2 \\ V_3 \end{array} \begin{array}{c} \begin{array}{cccc} I & A_1 & A_2 & A_3 \end{array} \\ \begin{pmatrix} 1 & \frac{q^2}{2} - q & \frac{q^2}{2} & q-2 \\ 1 & \frac{q}{2} & -\frac{q}{2} & -1 \\ 1 & -\frac{q}{2}+1 & -\frac{q}{2} & q-2 \\ 1 & -\frac{q}{2} & \frac{q}{2} & -1 \end{pmatrix} \end{array}$$

A_2, $A_1 + A_3$ are strongly regular

$A_1 + A_3$ has the same parameter as the symplectic polar graph

$q = 2^s$. $R_1 \cup R_3$: symplectic polar graph?

$O(3,q)$	$Sp(2s,2)$
$\mathbb{F}_q^3$	$\mathbb{F}_2^{2s} \cong \mathbb{F}_q^3/\langle\nu\rangle$
X	$\mathbb{F}_2^{2s} - \{0\}$
R_1	orthogonal
R_2	non-orthogonal
R_3	orthogonal

X: points not on $\mathcal{Q}$, not $\langle\nu\rangle$

$Q : \mathbb{F}_q^3 \to \mathbb{F}_q$, $f : \mathbb{F}_q^3 \times \mathbb{F}_q^3 \to \mathbb{F}_q$,

$f(x,y) = Q(x+y) + Q(x) + Q(y)$

$\overline{f} : \mathbb{F}_q^3/\langle\nu\rangle \times \mathbb{F}_q^3/\langle\nu\rangle \to \mathbb{F}_q$

$\mathrm{Tr} \circ \overline{f} : \mathbb{F}_q^3/\langle\nu\rangle \times \mathbb{F}_q^3/\langle\nu\rangle \to \mathbb{F}_2$

$$P = \begin{array}{c} \\ \mathbb{R}\mathbf{1} \\ V_1 \\ V_2 \\ V_3 \end{array} \begin{array}{c} \begin{array}{cccc} I & A_1 & A_2 & A_3 \end{array} \\ \begin{pmatrix} 1 & \frac{q^2}{2} - q & \frac{q^2}{2} & q-2 \\ 1 & \frac{q}{2} & -\frac{q}{2} & -1 \\ 1 & -\frac{q}{2}+1 & -\frac{q}{2} & q-2 \\ 1 & -\frac{q}{2} & \frac{q}{2} & -1 \end{pmatrix} \end{array}$$

R_3 is a union of K_{q-1}'s.

$R_1 \cup R_3$: symplectic polar graph?

R_3: union of K_{q-1}'s

$O(3,q)$	$Sp(2s,2)$	$Sp(2,q) = Sp(\overline{f})$
$\mathbb{F}_q^3$	$\mathbb{F}_2^{2s} \cong \mathbb{F}_q{}^3/\langle \nu \rangle$	$\mathbb{F}_q^2$
X	$\mathbb{F}_2^{2s} - \{0\}$	$\mathbb{F}_q^2 - \{0\}$
R_1	orthogonal	$\mathrm{Tr} \circ \overline{f}(a,b) = 0$
R_2	non-orthogonal	$\mathrm{Tr} \circ \overline{f}(a,b) \neq 0$
R_3	orthogonalsame $\mathbb{F}_q$-sp.	$\overline{f}(a,b) = 0$

X: points not on $\mathcal{Q}$, not $\langle \nu \rangle$

$Q : \mathbb{F}_q^3 \to \mathbb{F}_q$, $f : \mathbb{F}_q^3 \times \mathbb{F}_q^3 \to \mathbb{F}_q$

$f(x,y) = Q(x+y) + Q(x) + Q(y)$

$\overline{f} : \mathbb{F}_q^3/\langle \nu \rangle \times \mathbb{F}_q^3/\langle \nu \rangle \to \mathbb{F}_q$

$\mathrm{Tr} \circ \overline{f} : \mathbb{F}_q^3/\langle \nu \rangle \times \mathbb{F}_q^3/\langle \nu \rangle \to \mathbb{F}_2$

$$R_1 = \{(\langle x\rangle, \langle y\rangle) \mid \langle x, y\rangle : \text{ secant}\}$$
$$\stackrel{?}{=} \{(\langle x\rangle, \langle y\rangle) \mid \operatorname{Tr}\circ\overline{f}(\overline{x}, \overline{y}) = 0\}$$

$$\langle x, y\rangle : \text{ secant} \quad (Q(x) = Q(y) = 1)$$
$$\iff \#\{\langle u\rangle \in \mathcal{Q} \mid \langle u\rangle \subset \langle x, y\rangle\} = 2$$
$$\iff \#\{\alpha \in \mathbb{F}_q \mid Q(\alpha x + y) = 0\} = 2$$
$$\iff \#\{\alpha \in \mathbb{F}_q \mid \alpha^2 + f(x, y)\alpha + 1 = 0\} = 2$$
$$\iff \#\{\alpha \in \mathbb{F}_q \mid \alpha^2 + \overline{f}(\overline{x}, \overline{y})\alpha + 1 = 0\} = 2$$
$$\iff \operatorname{Tr}(\overline{f}(\overline{x}, \overline{y})^{-1}) = 0$$

$$(\langle x\rangle, \langle y\rangle) \in R_1 \cup R_3 : \text{ strongly regular}$$
$$\iff \mathrm{Tr}(\overline{f}(\overline{x}, \overline{y})^{-1}) = 0 \text{ or } \overline{f}(\overline{x}, \overline{y}) = 0$$
$$\iff \mathrm{Tr}(\overline{f}(\overline{x}, \overline{y})^{q-2}) = 0 \text{ or } \overline{f}(\overline{x}, \overline{y}) = 0$$
$$\iff \mathrm{Tr}(\overline{f}(\overline{x}, \overline{y})^{q-2}) = 0$$

What if we replace $q-2$ by an arbitrary integer e with $(e, q-1) = 1$?
Always strongly regular, non-isomorphic if $e \not\equiv 2^r e' \pmod{q-1}$ for $\forall r$ (In September, by computer experiments)
$\rightarrow$ deadline of abstract submission

A. M. $\to$ Bill Kantor (Nov. 16) in Tokyo:
$q = 2^s$, $(e, q-1) = 1$, $X = \mathbb{F}_q^2 - \{0\}$,
$f : \mathbb{F}_q^2 \times \mathbb{F}_q^2 \to \mathbb{F}_q$: alt. bil. form., $\Gamma^{(e)} = (X, R)$,
where

$$R = \{(x, y) \mid \mathrm{Tr}(f(x, y)^e) = 0\}$$

I would call $\Gamma^{(e)}$ a twisted symplectic polar graph.
Jackson–Wild (1997)
Also relevant: Gordon–Mills–Welch (1962),
Jackson (1993)

Kantor (2001) implies
$\Gamma^{(e)} \cong \Gamma^{(e')} \iff \exists r,\ e = 2^r e' \pmod{q-1}$

Back to the question of Eiichi Bannai (Nov. 18):

$$R = \{(x,y) \mid \mathrm{Tr}(f(x,y)^e) = 0\}$$

$\supset \{(x,y) \mid f(x,y) = 0\}$: union of K_{q-1}'s

: spread of Delsarte cliques

By Haemers–Tonchev (1996), $R = R_1 \cup R_3$, R_2: complement of R, 3-class association scheme.

Varying $e \implies$ non-isomorphic $R \implies$ non-isomorphic a.s.

Gordon–Mills–Welch (GMW) difference set

Ingredients:

- q_0: prime power
- $s \geq 2$
- D: difference set whose development is a design with the same parameters as $PG(s-1, q_0)$
- $m \geq 2$

Output: difference set whose development is a design with the same parameters as $PG(ms-1, q_0)$

Isomorphism determined by Jackson-Wild, Kantor

GMW	BCN $O(3,q)$	Twisted Sympl.
q_0	2	2
s	$q=2^s$	$q=2^s$
D	$\{a\in\mathbb{F}_q^*\mid \mathrm{Tr}\,a^{-1}=0\}$	$\{a\in\mathbb{F}_q^*\mid \mathrm{Tr}\,a^{e}=0\}$
m	2	2
$\downarrow$	$\downarrow$	$\downarrow$
diff. set.	srg	srg
$\searrow$	$\downarrow$	$\swarrow$

symmetric design with
same parameters as $PG(2s-1,2)$

srg+spread of Delsarte cliques $\rightarrow$ a.s.
Actually, BCN mentions the case $m>2$.

GMW	BCN $O(m+1,q)$	Twisted Sympl.
q_0	2	2
s	$q=2^s$	$q=2^s$
D	$\{a \in \mathbb{F}_q^* \mid \operatorname{Tr} a^{-1}=0\}$	$\{a \in \mathbb{F}_q^* \mid \operatorname{Tr} a^{e}=0\}$
m	m:even	m:even
$\downarrow$	$\downarrow$	$\downarrow$
diff. set.	srg	srg

However, for $m>2$, spread does not consist of Delsarte cliques

$\Longrightarrow$ cannot use Haemers-Tonchev to get 3-class association scheme

$s > 1$, $q = 2^s$, $m > 2$: even, $V = \mathbb{F}_q^m$,
$f : V \times V \to \mathbb{F}_q$: non degenerate alt. bil. form,
$X = V - \{0\}$

D : difference set in $\mathbb{F}_q^*$

$$R_1 = \{(x, y) \mid f(x, y) \in D\}$$
$$R_2 = \{(x, y) \mid f(x, y) \notin D,\ f(x, y) \neq 0\}$$
$$R_3 = \{(x, y) \mid \langle x\rangle_{\mathbb{F}_q} = \langle y\rangle_{\mathbb{F}_q}\}$$
$$R_4 = \{(x, y) \mid \langle x\rangle_{\mathbb{F}_q} \neq \langle y\rangle_{\mathbb{F}_q},\ f(x, y) = 0\}$$

R_2: strongly regular, $(X, \{R_i\}_{i=0}^{4})$: a.s.
BCN: $O(m + 1, q)$ by $D = \{a \in \mathbb{F}_q^* \mid \mathrm{Tr}\, a^{-1} = 0\}$
$m = 2 \implies R_4 = \emptyset$

GMW	BCN $O(m+1,q)$	Twisted Sympl.
q_0	2	2
s	$q = 2^s$	$q = 2^s$
D	$\{a \in \mathbb{F}_q^* \mid \mathrm{Tr}\, a^{-1} = 0\}$	$\{a \in \mathbb{F}_q^* \mid \mathrm{Tr}\, a^e = 0\}$
m	m:even	m:even
$\downarrow$	$\downarrow$	$\downarrow$
diff. set.	srg	srg

$s > 1$, $q = q_0^s$, m: even, $V = \mathbb{F}_q^m$, $f : V \times V \to \mathbb{F}_q$: non degenerate alt. bil. form,
$X = PG(V) = PG(ms-1, q_0)$ (over $\mathbb{F}_{q_0}$)

$$\bar{\ } : \mathbb{F}_q^* \to \mathbb{F}_q^*/\mathbb{F}_{q_0}^*$$

D : difference set in $\mathbb{F}_q^*/\mathbb{F}_{q_0}^* = PG(s-1, q_0)$

$$R_1 = \{([x],[y]) \mid \overline{f(x,y)} \in D\}$$

$$R_2 = \{([x],[y]) \mid \overline{f(x,y)} \notin D,\ f(x,y) \neq 0\}$$

$$R_3 = \{([x],[y]) \mid \langle x\rangle_{\mathbb{F}_q} = \langle y\rangle_{\mathbb{F}_q}\}$$

$$R_4 = \{([x],[y]) \mid \langle x\rangle_{\mathbb{F}_q} \neq \langle y\rangle_{\mathbb{F}_q},\ f(x,y) = 0\}$$

R_2: strongly regular, $(X, \{R_i\}_{i=0}^4)$: a.s. (?)
$R_4 = \emptyset$ if $m = 2$.

Mark Pankov,
Eiichi Bannai and Tatsuro Ito asked:
Twisted symplectic dual polar graphs?

Van Dam and Koolen (2005) introduced the twisted Grassmann graphs,
Munemasa–Tonchev (2011) showed that these graphs are the block graphs of designs

I don’t have any idea how to twist these.

Thank you for your attention.