# Grassmann Graphs

Akihiro Munemasa

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**Notation.**

$$\begin{bmatrix} W \\ i \end{bmatrix} : \text{the set of all } i\text{-dimensional subspaces of a vector space } W \text{ over } \mathbb{F}_q$$

$$\left\langle \begin{matrix} m, j \\ i \end{matrix} \right\rangle = (q^m - q^j)(q^m - q^{j+1}) \cdots (q^m - q^{j+i-1}) = q^{ij} \left\langle \begin{matrix} m-j, 0 \\ i \end{matrix} \right\rangle$$

$$\begin{bmatrix} m \\ i \end{bmatrix} = \left\langle \begin{matrix} m, 0 \\ i \end{matrix} \right\rangle \left\langle \begin{matrix} i, 0 \\ i \end{matrix} \right\rangle^{-1}$$

$\langle y_1, \dots, y_i \rangle$ : the linear span of $y_1, \dots, y_i \in V$

Let $V$ be an $n$-dimensional vector space over $\mathbb{F}_q$, and fix $X \in \begin{bmatrix} V \\ j \end{bmatrix}$.

1. Count $\{(Y, y_1, \dots, y_i) \mid Y = \langle y_1, \dots, y_i \rangle \in \begin{bmatrix} V \\ i \end{bmatrix}\}$ to derive $\#\begin{bmatrix} V \\ i \end{bmatrix} = \begin{bmatrix} n \\ i \end{bmatrix}$.

2. Count $\{(Y, y_1, \dots, y_i) \mid Y = \langle y_1, \dots, y_i \rangle \in \begin{bmatrix} V \\ i \end{bmatrix},\ X \cap Y = 0\}$ to derive $\#\{Y \in \begin{bmatrix} V \\ i \end{bmatrix} \mid X \cap Y = 0\} = q^{ij} \begin{bmatrix} n-j \\ i \end{bmatrix}$.

3. Count $\{(Y, y_{j+1}, \dots, y_i) \mid Y = \langle X, y_{j+1}, \dots, y_i \rangle \in \begin{bmatrix} V \\ i \end{bmatrix},\ X \subset Y\}$ to derive $\#\{Y \in \begin{bmatrix} V \\ i \end{bmatrix} \mid X \subset Y\} = \begin{bmatrix} n-j \\ i-j \end{bmatrix}$.

4. Count $\{(Y, W, y_{m+1}, \dots, y_i) \mid Y = \langle W, y_{m+1}, \dots, y_i \rangle \in \begin{bmatrix} V \\ i \end{bmatrix},\ X \cap Y = W \in \begin{bmatrix} X \\ m \end{bmatrix}\}$ to derive $\#\{Y \in \begin{bmatrix} V \\ i \end{bmatrix} \mid X \cap Y \in \begin{bmatrix} X \\ m \end{bmatrix}\} = q^{(i-m)(j-m)} \begin{bmatrix} j \\ m \end{bmatrix} \begin{bmatrix} n-j \\ i-m \end{bmatrix}$.

**Definition.** Let $n \geq 2d$. The **Grassmann graph** $J_q(n, d)$ is the graph with vertex set $\begin{bmatrix} V \\ d \end{bmatrix}$, where $V$ is a vector space of dimension $n$ over $\mathbb{F}_q$, and two vertices $X, Y$ are adjacent whenever $\dim X \cap Y = d - 1$.

Let $X, Y, Z \in \begin{bmatrix} V \\ d \end{bmatrix}$ and assume $\dim Y \cap Z = d - 1$. Then

$$\begin{aligned} \dim X \cap Y &\geq \dim X \cap Y \cap Z \\ &= \dim X \cap Z + \dim Y \cap Z - \dim(X \cap Z + Y \cap Z) \\ &\geq \dim X \cap Z + \dim Y \cap Z - \dim Z \\ &= \dim X \cap Z - 1. \end{aligned} \tag{1}$$

Thus $\dim X \cap Y = \dim X \cap Z - 1$ implies $X \cap Y \subset Z = X \cap Z + Y \cap Z$.

For $i = 0, 1, \dots, d$, we claim

$$\text{the distance between } X \text{ and } Y = d - \dim X \cap Y.$$

Indeed, if $\dim X \cap Y = d - i$, then one can explicitly construct a path of length $i$ from $X$ to $Y$, so the distance between $X$ and $Y$ is at most $i$. The reverse inequality can be proved by induction on the distance. Suppose $X$ and $Y$ are at distance $i \geq 2$. Choose $Z$ at distance $i-1$ from $X$, and distance 1 from $Y$. Then by induction, $\dim X \cap Z \geq d-(i-1)$. The inequality then follows from (1).

Define

$$R_i = \{(X,Y) \mid X, Y \in \begin{bmatrix} n \\ d \end{bmatrix},\ \dim X \cap Y = d-i\}.$$

5. $GL(V)$ acts transitively on $R_i$ for each $i = 0, 1, \ldots, d$.

6. $k_i = \#\{Y \mid (X,Y) \in R_i\} = q^{i^2} \begin{bmatrix} d \\ i \end{bmatrix} \begin{bmatrix} n-d \\ i \end{bmatrix}$.

Suppose $(X,Y) \in R_i$, $(X,Z) \in R_{i-1}$, $(Z,Y) \in R_1$. Then there is a bijection from $\{Z \mid (X,Z) \in R_{i-1},\ (Z,Y) \in R_1\}$ to the set

$$\{(X',Y') \mid X \cap Y \subset X' \in \begin{bmatrix} X \\ d-i+1 \end{bmatrix},\ X \cap Y \subset Y' \in \begin{bmatrix} Y \\ d-1 \end{bmatrix}\}$$

defined by

$$\phi : Z \mapsto (X \cap Z, Y \cap Z), \quad \psi : (X',Y') \mapsto X' + Y'.$$

Indeed, $\psi \circ \phi = \mathrm{id}$ since equality holds in (1). To show $\phi \circ \psi = \mathrm{id}$, we need $X \cap (X'+Y') = X'$ and $Y \cap (X'+Y') = Y'$ whenever $X \cap Y \subset X'$ and $X \cap Y \subset Y'$. If $x = x'+y' \in X \cap (X'+Y')$, then $y' \in (X+X') \cap Y' = X \cap Y' \subset X \cap Y \subset X'$, so $x = x'+y' \in X'$. This proves $X \cap (X'+Y') \subset X'$, while the reverse containment is obvious. Similarly, one obtains $Y \cap (X'+Y') = Y'$.

Now

$$\begin{aligned}
c_i &= \#\{Z \mid (X,Z) \in R_{i-1},\ (Z,Y) \in R_1\} \\
&= \#\{(X',Y') \mid X \cap Y \subset X' \in \begin{bmatrix} X \\ d-i+1 \end{bmatrix},\ X \cap Y \subset Y' \in \begin{bmatrix} Y \\ d-1 \end{bmatrix}\} \\
&= \#\{X' \in \begin{bmatrix} X \\ d-i+1 \end{bmatrix} \mid X \cap Y \subset X'\} \times \#\{Y' \in \begin{bmatrix} Y \\ d-1 \end{bmatrix} \mid X \cap Y \subset Y'\} \\
&= \begin{bmatrix} i \\ 1 \end{bmatrix} \begin{bmatrix} i \\ i-1 \end{bmatrix} \\
&= \begin{bmatrix} i \\ 1 \end{bmatrix}^2.
\end{aligned}$$

Finally,

$$b_i = \frac{c_{i+1} k_{i+1}}{k_i} = q^{2i+1} \begin{bmatrix} d-i \\ 1 \end{bmatrix} \begin{bmatrix} n-d-i \\ 1 \end{bmatrix}.$$