Godsil–McKay switching and twisted Grassmann graphs

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8SHCC

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$\downarrow$ Godsil–McKay switching

twisted Grassmann graph

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The resulting graph is the twisted Grassmann graph $\tilde{J}_q(2d+1, d+1)$.

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- Seidel switching
- Doob graphs
- Godsil–McKay switching (1982)
- DRG, Grassmann graphs

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2014+ distorted$\leftrightarrow$Godsil–McKay switching

$$\begin{array}{ccc} \mathrm{PG}_{d}(2d, q) & \xrightarrow{\text{block graph}} & J_q(2d+1, d+1) \\ \text{distort} \Big\downarrow & & \Big\downarrow \\ \text{new design} & \xrightarrow{\text{block graph}} & \tilde{J}_q(2d+1, d+1) \end{array}$$

Block graph = graph with blocks as vertices, adjacent iff intersect at maximal size.

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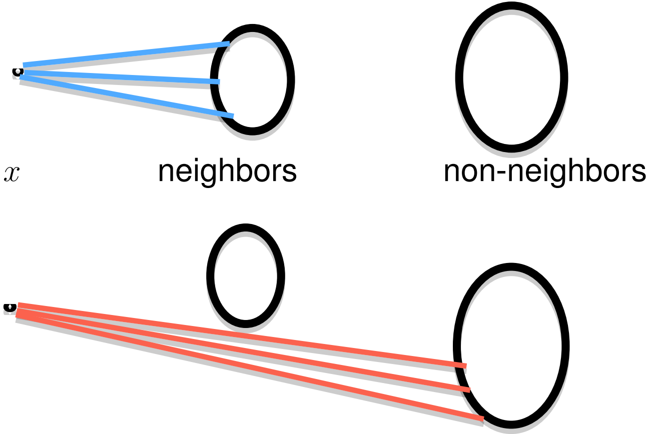
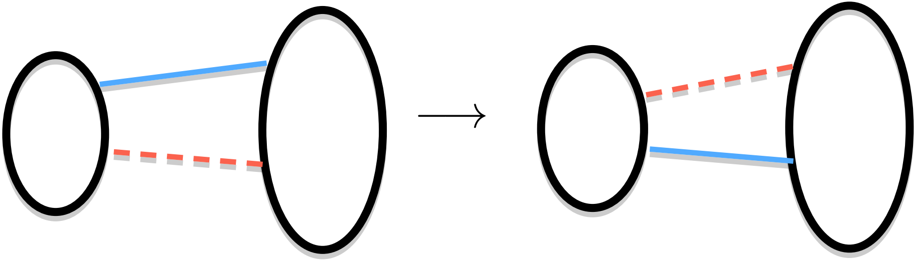
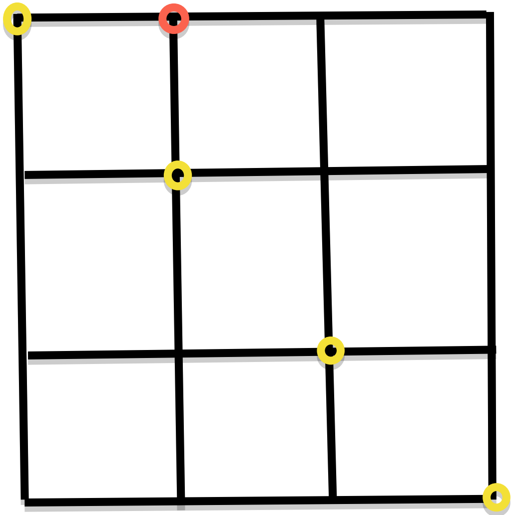
- The original definition of $\tilde{J}_q(2d+1, d+1)$ does not use a polarity.
- Both distorting and GM switching rely on a polarity.

Seidel switching

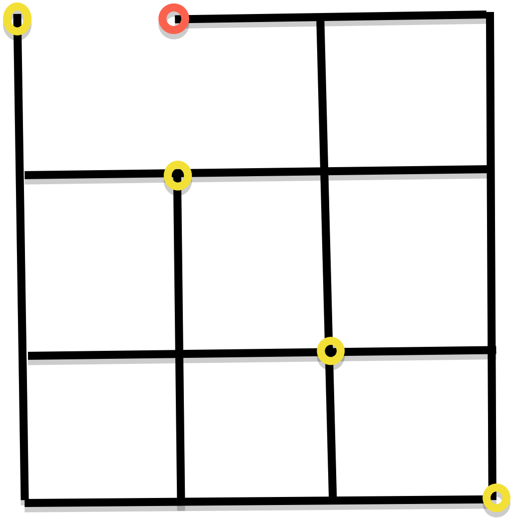


Seidel switching (II)





$\mathrm{SRG}(16,6,2,2)$: $K_4 \times K_4 \ncong$ Shrikhande graph



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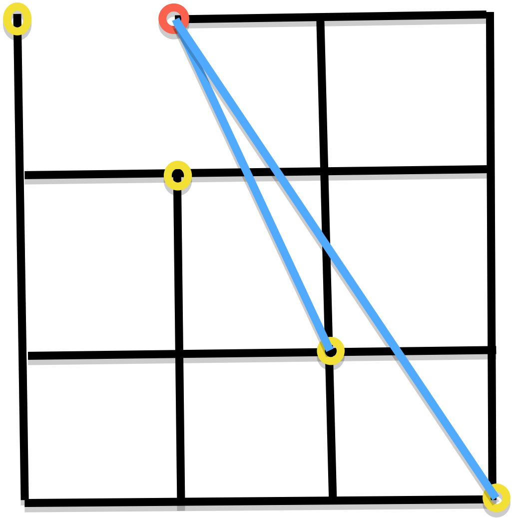

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with respect to $C = \{(x, x) \mid x \in K_4\}$.

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$$((x, x), j) \sim ((x, y), j) \mapsto ((x, x), j) \not\sim ((x, y), j)$$

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$$D \ni ((x, y), j) \begin{array}{l} \sim ((x, x), j) \in C_j \\ \sim ((y, y), j) \in C_j \\ \not\sim ((z, z), j) \in C_j \\ \not\sim ((w, w), j) \in C_j \end{array} \longrightarrow \begin{array}{l} \not\sim ((x, x), j) \in C_j \\ \not\sim ((y, y), j) \in C_j \\ \sim ((z, z), j) \in C_j \\ \sim ((w, w), j) \in C_j \end{array}$$

$\Gamma = (X, E)$: graph, $X = D \cup (\bigcup_i C_i)$.
Assume $\forall x \in D$, $\forall i$, x is adjacent to $0, 1/2$ or all vertices of C_i.
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Theorem (Godsil–McKay, 1982)

If $\{C_i\}_i$ is *equitable*, then the resulting graph is cospectral with the original.

Equitable: $\forall i$, $\forall x \in C_i, \forall y \in C_i$, $\forall j$,
$|\Gamma(x) \cap C_j| = |\Gamma(y) \cap C_j|$.

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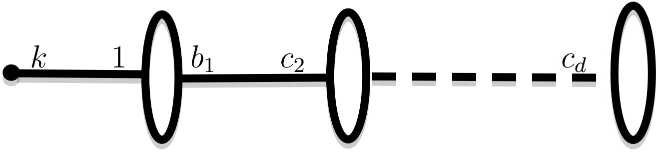
$D \ni ((x,y),j)$ is adjacent to 2 out of 4 vertices of C_j,
$D \ni ((x,y),j)$ is adjacent to 0 vertices of $C_{j'}$, $j' \neq j$.

Distance-Regular Graphs

A connected graph of diameter d is called a distance-regular graph if $\exists\{k, b_1, \ldots, b_{d-1}; 1, c_2, \ldots, c_d\}$ such that



Examples with unbounded d:

- $H(n,q) = K_q^n$, $J(v,d)$, $J_q(v,d)$, dual polar graphs, forms graphs
- halved, folded graphs of above
- Doob, Hemmeter, Ustimenko graphs

Johnson graph $J(v, k)$

- $|V| = v$
- $\binom{V}{k}$ = the collection of k-subsets of V
- $W_1 \sim W_2 \iff |W_1 \cap W_2| = k - 1$.

Then $J(v, k) \cong J(v, v - k)$.

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Vector space analogue?

Grassmann graph $J_q(v, d)$

- $V =$ vector space over $\mathrm{GF}(q)$, $\dim V = v$
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Theorem (Metsch (1995))

$J_q(v, d)$ is characterized uniquely by the intersection array except

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2. $v = 2d$, $v = 2d + 1$
3. $v = 2d + 2$, $q = 2, 3$
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We focus on $J_q(2d + 1, d) \cong J_q(2d + 1, d + 1)$.

Twisted Grassmann graph $\tilde{J}_q(2d+1, d+1)$

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Instead of modifying the vertex set, can we switch edges?

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$$P = \begin{bmatrix} V \\ 1 \end{bmatrix} \quad \text{projective points}$$

$(P, \begin{bmatrix} V \\ d+1 \end{bmatrix})$: 2-design, with incidence $p \sim W \iff p \subset W$.

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Theorem (M.–Tonchev (2011))

The block graph of the distorted design $\cong \tilde{J}_q(2d+1, d+1)$.

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