On the smallest eigenvalues of the line graphs of some trees

Akihiro (昭弘) Munemasa (宗政)
Tohoku University (東北大学)

Joint work with Yoshio Sano and Tetsuji Taniguchi

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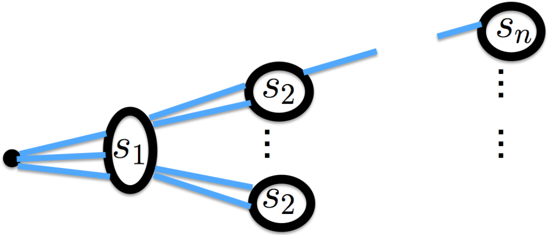
Let T be the tree depicted below:



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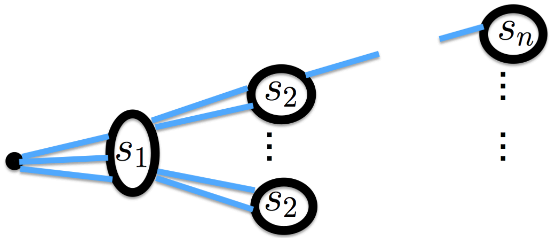

Then the smallest eigenvalue of the line graph of T is the smallest zero of the polynomial $g_n(x)$, where

$$g_0(x) = 1,\ g_1(x) = x + 1,$$
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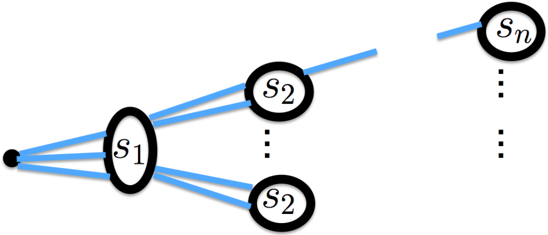

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In particular, it is independent of s_1.

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where X is a $|V| \times m$ matrix, giving a representation:

$$(u, v) = \begin{cases} -\lambda_{\min}(\Gamma) & \text{if } u = v, \\ 1 & \text{if } u \sim v, \\ 0 & \text{otherwise.} \end{cases}$$

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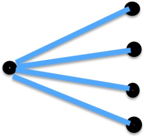
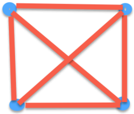


s-claw



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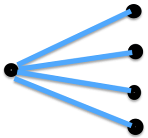

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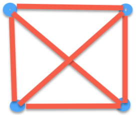

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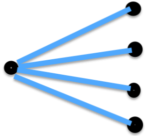

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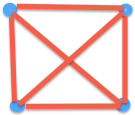

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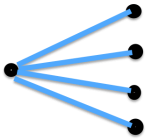

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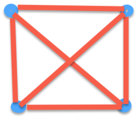

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Conversely, all graphs Δ with $\lambda_{\min}(\Delta) \geq -2$ are essentially known (generalized line graphs + finitely many exceptions).

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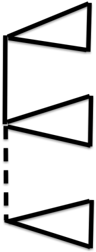
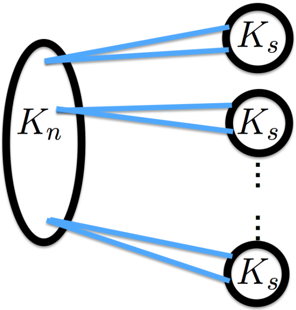
$\lambda_{\min}($  $) = -\sqrt{3}$

Cvetković–Stevanović (2003): $\lambda_{\min}$ of the following graph is

$$\frac{s-2-\sqrt{s(s+4)}}{2}$$



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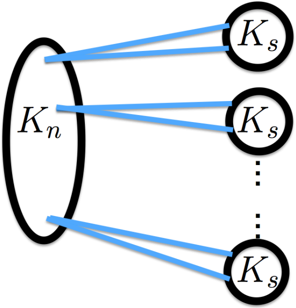

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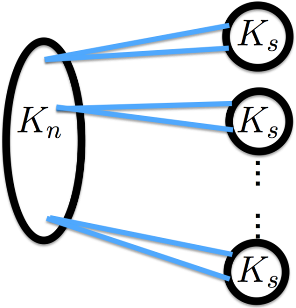

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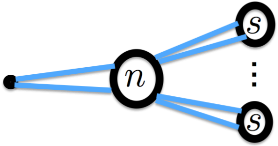
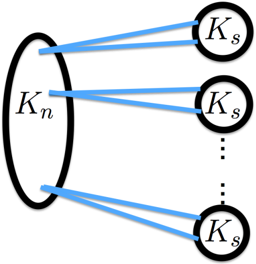
The spectrum can be obtained by a formula of Schwenk (1973).

The corona $K_n \otimes K_s$ of K_n and K_s is the line graph of the tree





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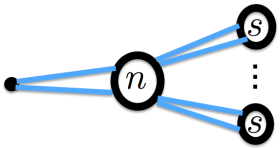

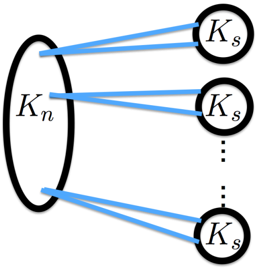

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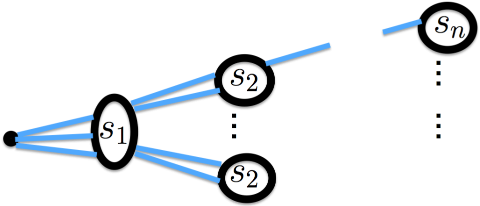
Cvetković–Stevanović (2003): are there other family of line graphs of trees with constant smallest eigenvalue?

Let $T_{s_1,s_2,\ldots,s_n}$ be the tree depicted below:



Then $\lambda_{\min}(L(T_{s_1,s_2,\ldots,s_n}))$ is the smallest zero of the polynomial $g_n(x)$, where

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In particular, it is independent of s_1.

The characteristic polynomial of $L(T_{s_1,s_2,\dots,s_n})$ is

$$\frac{1}{x+2}(g_{n+1}(x)+g_n(x))\prod_{i=1}^{n} g_i(x)^{\sigma_{n-i+1}}$$

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Thank you.