Generalized tensor products and related constructions

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Outline

- Tensor product
- Diță's construction and generalized tensor product
- Weaving
- Generalized tensor product as weaving with respect to J
- Weaving as generalized tensor product of variable-order matrices
- Generalized tensor product as a principal submatrix of strong Kronecker product

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References:

- R. Craigen, Ph.D Thesis (1991), JCD (1995)
- R. Hosoya and H. Suzuki, JACO (2003)
- J. Seberry and X.-M. Zhang, Australasian J. (1991)

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- A complex Hadamard matrix of order n is an $n \times n$ matrix H with entries in $\{\xi \in \mathbb{C} \mid |\xi| = 1\}$, satisfying $HH^* = nI$, where $*$ means the conjugate transpose.

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- A (real) Hadamard matrix of order n is an $n \times n$ matrix H with entries ± 1, satisfying $HH^\top = nI$.
- A weighing matrix of order n and weight k (denoted $W(n,k)$) is an $n \times n$ matrix W with entries in $\{0, \pm 1\}$, satisfying $WW^\top = kI$.
- A complex Hadamard matrix of order n is an $n \times n$ matrix H with entries in $\{\xi \in \mathbb{C} \mid |\xi| = 1\}$, satisfying $HH^* = nI$, where $*$ means the conjugate transpose.
- A type II (or inverse-orthogonal) matrix of order n is an $n \times n$ matrix H with nonzero complex entries, satisfying $HH^{(-)\top} = nI$, where $(-)$ means the entrywise inverse.

The tensor (Kronecker) product of two matrices H and K is

$$H \otimes K = \begin{bmatrix} H_{11}K & H_{12}K & \cdots & H_{1n}K \\ \vdots & \vdots & & \vdots \\ H_{i1}K & H_{i2}K & \cdots & H_{in}K \\ \vdots & \vdots & & \vdots \\ H_{j1}K & H_{j2}K & \cdots & H_{jn}K \\ \vdots & \vdots & & \vdots \\ H_{n1}K & H_{n2}K & \cdots & H_{nn}K \end{bmatrix}$$

Proposition

If H and K are Hadamard matrices of order n and m, respectively, then $H \otimes K$ is a Hadamard matrix of order nm.

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Proof.

$$\begin{aligned} & H_{i1}KK^\top H_{j1} + H_{i2}KK^\top H_{j2} + \cdots \\ &= (H_{i1}H_{j1} + H_{i2}H_{j2} + \cdots)mI \\ &= (HH^\top)_{ij}mI \\ &= \delta_{ij}nmI. \end{aligned}$$

□

$$H \otimes K = \begin{bmatrix} H_{11}K & H_{12}K & \cdots & H_{1n}K \\ H_{21}K & H_{22}K & \cdots & H_{2n}K \\ \vdots & \vdots & & \vdots \\ H_{n1}K & H_{n2}K & \cdots & H_{nn}K \end{bmatrix}$$

Proof.

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□

$$
\begin{bmatrix}
H_{11}K_1 & H_{12}K_2 & \cdots & H_{1n}K_n \\
H_{21}K_1 & H_{22}K_2 & \cdots & H_{2n}K_n \\
\vdots & \vdots & & \vdots \\
H_{n1}K_1 & H_{n2}K_2 & \cdots & H_{nn}K_n
\end{bmatrix}
$$

Proof.

$$
\begin{aligned}
& H_{i1}K_1K_1^\top H_{j1} + H_{i2}K_2K_2^\top H_{j2} + \cdots \\
& = (H_{i1}H_{j1} + H_{i2}H_{j2} + \cdots)mI \\
& = (HH^\top)_{ij}mI \\
& = \delta_{ij}mnI.
\end{aligned}
$$

□

Proposition (Diță's construction)

If H is a Hadamard matrix of order n, and $K_1, \dots, K_n$ are Hadamard matrices of order m, then

$$H \otimes (K_1, \dots, K_n) = \begin{bmatrix} H_{11}K_1 & H_{12}K_2 & \cdots & H_{1n}K_n \\ H_{21}K_1 & H_{22}K_2 & \cdots & H_{2n}K_n \\ \vdots & \vdots & & \vdots \\ H_{n1}K_1 & H_{n2}K_2 & \cdots & H_{nn}K_n \end{bmatrix}$$

is a Hadamard matrix of order nm.

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is a Hadamard matrix of order nm.

Proposition (Diță's construction)

If H is a complex Hadamard matrix of order n, and $K_1, \ldots, K_n$ are complex Hadamard matrices of order m, then

$$H \otimes (K_1, \ldots, K_n) = \begin{bmatrix} H_{11}K_1 & H_{12}K_2 & \cdots & H_{1n}K_n \\ H_{21}K_1 & H_{22}K_2 & \cdots & H_{2n}K_n \\ \vdots & \vdots & & \vdots \\ H_{n1}K_1 & H_{n2}K_2 & \cdots & H_{nn}K_n \end{bmatrix}$$

is a complex Hadamard matrix of order nm.

Proposition (Diță's construction)

If H is an inverse-orthogonal matrix of order n, and $K_1, \ldots, K_n$ are inverse-orthogonal matrices of order m, then

$$H \otimes (K_1, \ldots, K_n) = \begin{bmatrix} H_{11}K_1 & H_{12}K_2 & \cdots & H_{1n}K_n \\ H_{21}K_1 & H_{22}K_2 & \cdots & H_{2n}K_n \\ \vdots & \vdots & & \vdots \\ H_{n1}K_1 & H_{n2}K_2 & \cdots & H_{nn}K_n \end{bmatrix}$$

is an inverse-orthogonal matrix of order nm.

Proposition (Diță's construction)

If H is an inverse-orthogonal matrix of order n, and $K_1, \ldots, K_n$ are inverse-orthogonal matrices of order m, then

$$H \otimes (K_1, \ldots, K_n) = \begin{bmatrix} H_{11}K_1 & H_{12}K_2 & \cdots & H_{1n}K_n \\ H_{21}K_1 & H_{22}K_2 & \cdots & H_{2n}K_n \\ \vdots & \vdots & & \vdots \\ H_{n1}K_1 & H_{n2}K_2 & \cdots & H_{nn}K_n \end{bmatrix}$$

is an inverse-orthogonal matrix of order nm.

Not only K but also H can be replaced by $H_1, \ldots, H_m$.

Definition (generalized tensor product)

$$H_1, \ldots, H_m : \text{matrices of order } n,$$
$$K_1, \ldots, K_n : \text{matrices of order } m.$$

Let Δ_{ij} be the diagonal matrix defined by

$$(\Delta_{ij})_{hh} = (H_h)_{ij} \quad (1 \le i, j \le m,\ 1 \le h \le n)$$

The generalized tensor product is

$$(H_1, \ldots, H_m) \otimes (K_1, \ldots, K_n) = \begin{bmatrix} \Delta_{11}K_1 & \Delta_{12}K_2 & \cdots & \Delta_{1n}K_n \\ \Delta_{21}K_1 & \Delta_{22}K_2 & \cdots & \Delta_{2n}K_n \\ \vdots & \vdots & & \vdots \\ \Delta_{n1}K_1 & \Delta_{n2}K_2 & \cdots & \Delta_{nn}K_n \end{bmatrix}$$

Proposition (Hosoya and Suzuki, 2003)

$$H_1, \ldots, H_m : \text{matrices of order } n,$$
$$K_1, \ldots, K_n : \text{matrices of order } m,$$
$$(\Delta_{ij})_{hh} = (H_h)_{ij} \quad (1 \leq i, j \leq m,\ 1 \leq h \leq n)$$

Then the generalized tensor product

$$(H_1, \ldots, H_m) \otimes (K_1, \ldots, K_n) = \begin{bmatrix} \Delta_{11}K_1 & \Delta_{12}K_2 & \cdots & \Delta_{1n}K_n \\ \Delta_{21}K_1 & \Delta_{22}K_2 & \cdots & \Delta_{2n}K_n \\ \vdots & \vdots & & \vdots \\ \Delta_{n1}K_1 & \Delta_{n2}K_2 & \cdots & \Delta_{nn}K_n \end{bmatrix}$$

is an inverse-orthogonal matrix of order mn if and only if $H_1, \ldots, H_m, K_1, \ldots, K_n$ are inverse-orthogonal matrices.

The method of weaving

$$
\begin{aligned}
M &: m \times n \ (0,1)\text{-matrix}, & \\
r_i &: \text{row sum of } M & (1 \le i \le m), \\
A_i &: r_i \times r_i \text{ matrix} & (1 \le i \le m), \\
c_j &: \text{column sum of } M & (1 \le j \le n), \\
B_j &: c_j \times c_j \text{ matrix} & (1 \le j \le n)
\end{aligned}
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Set

$$N = \sum_{i=1}^{m} r_i = \sum_{j=1}^{n} c_j.$$

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B_j &: c_j \times c_j \text{ matrix} && (1 \le j \le n)
\end{aligned}
$$

Set

$$N = \sum_{i=1}^{m} r_i = \sum_{j=1}^{n} c_j.$$

The method of weaving gives a weighing matrix $W(N, ab)$ provided

$$
\begin{aligned}
A_i &: W(r_i, a) && (1 \le i \le m), \\
B_j &: W(c_j, b) && (1 \le j \le n).
\end{aligned}
$$

$$M = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \color{red}{3} \\ \color{red}{2} \\ \color{red}{2} \end{matrix}$$

$$\begin{matrix} \color{red}{3} & \color{red}{1} & \color{red}{1} & \color{red}{2} \end{matrix}$$

$$M = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} 3 \\ 2 \\ 2 \end{matrix}$$

$$\begin{matrix} 3 & 1 & 1 & 2 \end{matrix}$$

		B_1	B_2	B_3	B_4
		$[3 \times 3]$	$[1 \times 1]$	$[1 \times 1]$	$[2 \times 2]$
A_1	$[3 \times 3]$				
A_2	$[2 \times 2]$				
A_3	$[2 \times 2]$				

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		$[3 \times 3]$	$[1 \times 1]$	$[1 \times 1]$	$[2 \times 2]$
A_1	$[3 \times 3]$				
A_2	$[2 \times 2]$				
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		B_1 $[3 \times 3]$	B_2 $[1 \times 1]$	B_3 $[1 \times 1]$	B_4 $[2 \times 2]$
A_1	$[3 \times 3]$			O	
A_2	$[2 \times 2]$		O		O
A_3	$[2 \times 2]$		O	O	

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		B_1	B_2	B_3	B_4
		$[3 \times 3]$	$[1 \times 1]$	$[1 \times 1]$	$[2 \times 2]$
A_1	$[3 \times 3]$	3×3	3×1	O	3×2
A_2	$[2 \times 2]$	2×3	O	2×1	O
A_3	$[2 \times 2]$	2×3	O	O	2×2

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$$A_i : r_i \times r_i,\ B_j : c_j \times c_j \longrightarrow r_i \times c_j \text{ matrix}$$

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		$[3 \times 3]$	$[1 \times 1]$	$[1 \times 1]$	$[2 \times 2]$
A_1	$[3 \times 3]$	3×3	3×1	O	3×2
A_2	$[2 \times 2]$	2×3	O	2×1	O
A_3	$[2 \times 2]$	2×3	O	O	2×2

$$A_i : r_i \times r_i,\ B_j : c_j \times c_j \longrightarrow r_i \times c_j \text{ matrix}$$

$$A_1 : 3 \times 3,\ B_4 : 2 \times 2 \longrightarrow 3 \times 2 \text{ matrix } (A_1 \mathbf{e}_3)(\mathbf{e}_1^\top B_4)$$

$$M = \begin{bmatrix} 1 & 1 & 0 & \color{red}{1} \\ 1 & 0 & \color{red}{1} & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} 3 \\ 2 \\ 2 \end{matrix}$$

$$\begin{matrix} 3 & 1 & 1 & 2 \end{matrix}$$

		B_1	B_2	B_3	B_4
		$[3 \times 3]$	$[1 \times 1]$	$[1 \times 1]$	$[2 \times 2]$
A_1	$[3 \times 3]$	$A_1 \mathbf{e}_1 \mathbf{e}_1^\top B_1$	$A_1 \mathbf{e}_2 \mathbf{e}_1^\top B_2$	O	$A_1 \color{red}{\mathbf{e}_3} \mathbf{e}_1^\top B_4$
A_2	$[2 \times 2]$	$A_2 \mathbf{e}_1 \mathbf{e}_2^\top B_1$	O	$A_2 \mathbf{e}_2 \color{red}{\mathbf{e}_1}^\top B_3$	O
A_3	$[2 \times 2]$	$A_3 \mathbf{e}_1 \mathbf{e}_3^\top B_1$	O	O	$A_3 \mathbf{e}_2 \mathbf{e}_2^\top B_4$

Weaving with respect to $M = J$

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{matrix} 4 \\ 4 \\ 4 \end{matrix}$$

$$\begin{matrix} 3 & 3 & 3 & 3 \end{matrix}$$

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$$\begin{matrix} 3 & 3 & 3 & 3 \end{matrix}$$

$$\begin{array}{llcccc} & & B_1 & B_2 & B_3 & B_4 \\ & & [3 \times 3] & [3 \times 3] & [3 \times 3] & [3 \times 3] \\ A_1 & [4 \times 4] & A_1\mathbf{e}_1\mathbf{e}_1^\top B_1 & A_1\mathbf{e}_2\mathbf{e}_1^\top B_2 & A_1\mathbf{e}_3\mathbf{e}_1^\top B_3 & A_1\mathbf{e}_4\mathbf{e}_1^\top B_4 \\ A_2 & [4 \times 4] & A_2\mathbf{e}_1\mathbf{e}_2^\top B_1 & A_2\mathbf{e}_2\mathbf{e}_2^\top B_2 & A_2\mathbf{e}_3\mathbf{e}_2^\top B_3 & A_2\mathbf{e}_4\mathbf{e}_2^\top B_4 \\ A_3 & [4 \times 4] & A_3\mathbf{e}_1\mathbf{e}_3^\top B_1 & A_3\mathbf{e}_2\mathbf{e}_3^\top B_2 & A_3\mathbf{e}_3\mathbf{e}_3^\top B_3 & A_3\mathbf{e}_4\mathbf{e}_3^\top B_4 \end{array}$$

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$$M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{matrix} 4 \\ 4 \\ 4 \end{matrix}$$

$$\begin{matrix} 3 & 3 & 3 & 3 \end{matrix}$$

$$\begin{array}{llcccc}
 & & B_1 & B_2 & B_3 & B_4 \\
 & & [3 \times 3] & [3 \times 3] & [3 \times 3] & [3 \times 3] \\
A_1 & [4 \times 4] & A_1 \mathbf{e}_1 \mathbf{e}_1^\top B_1 & A_1 \mathbf{e}_2 \mathbf{e}_1^\top B_2 & A_1 \mathbf{e}_3 \mathbf{e}_1^\top B_3 & A_1 \mathbf{e}_4 \mathbf{e}_1^\top B_4 \\
A_2 & [4 \times 4] & A_2 \mathbf{e}_1 \mathbf{e}_2^\top B_1 & A_2 \mathbf{e}_2 \mathbf{e}_2^\top B_2 & A_2 \mathbf{e}_3 \mathbf{e}_2^\top B_3 & A_2 \mathbf{e}_4 \mathbf{e}_2^\top B_4 \\
A_3 & [4 \times 4] & A_3 \mathbf{e}_1 \mathbf{e}_3^\top B_1 & A_3 \mathbf{e}_2 \mathbf{e}_3^\top B_2 & A_3 \mathbf{e}_3 \mathbf{e}_3^\top B_3 & A_3 \mathbf{e}_4 \mathbf{e}_3^\top B_4
\end{array}$$

$$[A_i \mathbf{e}_j \mathbf{e}_i^\top B_j] = [A_i E_{ji} B_j]$$

Weaving with respect to $M = J$

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{matrix} 4 \\ 4 \\ 4 \end{matrix}$$
$$\begin{matrix} 3 & 3 & 3 & 3 \end{matrix}$$

$$\begin{array}{cccccc}
 & & B_1 & B_2 & B_3 & B_4 \\
 & & [3 \times 3] & [3 \times 3] & [3 \times 3] & [3 \times 3] \\
A_1 & [4 \times 4] & A_1 \mathbf{e}_1 \mathbf{e}_1^\top B_1 & A_1 \mathbf{e}_2 \mathbf{e}_1^\top B_2 & A_1 \mathbf{e}_3 \mathbf{e}_1^\top B_3 & A_1 \mathbf{e}_4 \mathbf{e}_1^\top B_4 \\
A_2 & [4 \times 4] & A_2 \mathbf{e}_1 \mathbf{e}_2^\top B_1 & A_2 \mathbf{e}_2 \mathbf{e}_2^\top B_2 & A_2 \mathbf{e}_3 \mathbf{e}_2^\top B_3 & A_2 \mathbf{e}_4 \mathbf{e}_2^\top B_4 \\
A_3 & [4 \times 4] & A_3 \mathbf{e}_1 \mathbf{e}_3^\top B_1 & A_3 \mathbf{e}_2 \mathbf{e}_3^\top B_2 & A_3 \mathbf{e}_3 \mathbf{e}_3^\top B_3 & A_3 \mathbf{e}_4 \mathbf{e}_3^\top B_4
\end{array}$$

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Weaving with respect to $M = J$

$$
\begin{array}{llllll}
 & & B_1 & B_2 & B_3 & B_4 \\
 & & [3 \times 3] & [3 \times 3] & [3 \times 3] & [3 \times 3] \\
A_1 & [4 \times 4] & A_1 \mathbf{e}_1 \mathbf{e}_1^\top {\color{red} B_1} & A_1 \mathbf{e}_2 \mathbf{e}_1^\top {\color{red} B_2} & A_1 \mathbf{e}_3 \mathbf{e}_1^\top {\color{red} B_3} & A_1 \mathbf{e}_4 \mathbf{e}_1^\top {\color{red} B_4} \\
A_2 & [4 \times 4] & A_2 \mathbf{e}_1 \mathbf{e}_2^\top {\color{red} B_1} & A_2 \mathbf{e}_2 \mathbf{e}_2^\top {\color{red} B_2} & A_2 \mathbf{e}_3 \mathbf{e}_2^\top {\color{red} B_3} & A_2 \mathbf{e}_4 \mathbf{e}_2^\top {\color{red} B_4} \\
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\end{array}
$$

$$[A_i \mathbf{e}_j \mathbf{e}_i^\top B_j] = [A_i E_{ji} B_j]$$

Weaving with respect to $M = J$

$$
\begin{array}{llllll}
 & & B_1 & B_2 & B_3 & B_4 \\
 & & [3 \times 3] & [3 \times 3] & [3 \times 3] & [3 \times 3] \\
A_1 & [4 \times 4] & A_1\mathbf{e}_1\mathbf{e}_1^\top B_1 & A_1\mathbf{e}_2\mathbf{e}_1^\top B_2 & A_1\mathbf{e}_3\mathbf{e}_1^\top B_3 & A_1\mathbf{e}_4\mathbf{e}_1^\top B_4 \\
A_2 & [4 \times 4] & A_2\mathbf{e}_1\mathbf{e}_2^\top B_1 & A_2\mathbf{e}_2\mathbf{e}_2^\top B_2 & A_2\mathbf{e}_3\mathbf{e}_2^\top B_3 & A_2\mathbf{e}_4\mathbf{e}_2^\top B_4 \\
A_3 & [4 \times 4] & A_3\mathbf{e}_1\mathbf{e}_3^\top B_1 & A_3\mathbf{e}_2\mathbf{e}_3^\top B_2 & A_3\mathbf{e}_3\mathbf{e}_3^\top B_3 & A_3\mathbf{e}_4\mathbf{e}_3^\top B_4
\end{array}
$$

$$[A_i\mathbf{e}_j\mathbf{e}_i^\top B_j] = [A_i E_{ji} B_j]$$

$$
\begin{bmatrix}
\Delta_{11} K_1 & \Delta_{12} K_2 & \cdots & \Delta_{1n} K_n \\
\Delta_{21} K_1 & \Delta_{22} K_2 & \cdots & \Delta_{2n} K_n \\
\vdots & \vdots & & \vdots \\
\Delta_{n1} K_1 & \Delta_{n2} K_2 & \cdots & \Delta_{nn} K_n
\end{bmatrix} = [\Delta_{ij} K_j]
$$

$(\Delta_{ij})_{hh} = (A_h)_{ij}$

$$\begin{bmatrix} A_1\mathbf{e}_1\mathbf{e}_1^\top B_1 & \cdots \\ A_2\mathbf{e}_1\mathbf{e}_2^\top B_1 & \cdots \\ A_3\mathbf{e}_1\mathbf{e}_3^\top B_1 & \cdots \end{bmatrix}$$

$$= \begin{bmatrix} (A_1)_{11}\mathbf{e}_1^\top B_1 & \cdots \\ (A_1)_{21}\mathbf{e}_1^\top B_1 & \cdots \\ (A_1)_{31}\mathbf{e}_1^\top B_1 & \cdots \\ (A_2)_{11}\mathbf{e}_2^\top B_1 & \cdots \\ (A_2)_{21}\mathbf{e}_2^\top B_1 & \cdots \\ (A_2)_{31}\mathbf{e}_2^\top B_1 & \cdots \\ (A_3)_{11}\mathbf{e}_3^\top B_1 & \cdots \\ (A_3)_{21}\mathbf{e}_3^\top B_1 & \cdots \\ (A_3)_{31}\mathbf{e}_3^\top B_1 & \cdots \end{bmatrix} \cong \begin{bmatrix} (A_1)_{11}\mathbf{e}_1^\top B_1 & \\ (A_2)_{11}\mathbf{e}_2^\top B_1 & = \Delta_{11}B_1 \\ (A_3)_{11}\mathbf{e}_3^\top B_1 & \\ (A_1)_{21}\mathbf{e}_1^\top B_1 & \\ (A_2)_{21}\mathbf{e}_2^\top B_1 & = \Delta_{21}B_1 \\ (A_3)_{21}\mathbf{e}_3^\top B_1 & \\ (A_1)_{31}\mathbf{e}_1^\top B_1 & \\ (A_2)_{31}\mathbf{e}_2^\top B_1 & = \Delta_{31}B_1 \\ (A_3)_{31}\mathbf{e}_3^\top B_1 & \end{bmatrix}$$

Weaving with respect to $M = J$ and generalized tensor product

- $M = J_{m \times n}$
- $A_1, \ldots, A_m$: $n \times n$
- $B_1, \ldots, B_n$: $m \times m$

Then the weaving of $(A_1, \ldots, A_m)$ and $(B_1, \ldots, B_n)$ with respect to $M = J_{m \times n}$ is

$$[A_i E_{ji} B_j]$$

which coincides with the generalized tensor product

$$(A_1, \ldots, A_m) \otimes (B_1, \ldots, B_n) = [\Delta_{ij} B_j] \quad \text{where } (\Delta_{ij})_{hh} = (A_h)_{ij}$$

after appropriate row permutation

Definition (weaving)

M: $m \times n$ $(0,1)$-matrix.

$$
\begin{aligned}
&R_i = \{j \mid 1 \le j \le n,\ M_{ij} = 1\}, \quad r_i = |R_i|,\\
&\rho_i : R_i \to \{1, \dots, r_i\} \text{ bijection},\\
&A_i : r_i \times r_i \text{ matrix},\\
&C_j = \{i \mid 1 \le i \le m,\ M_{ij} = 1\}, \quad c_j = |C_j|,\\
&\gamma_j : C_j \to \{1, \dots, c_j\} \text{ bijection},\\
&B_j : c_j \times c_j \text{ matrix}.
\end{aligned}
$$

The weaving of $(A_1, \dots, A_m)$ and $(B_1, \dots, B_n)$ with respect to M is defined to be the $m \times n$ block matrix whose (i,j) block is the $r_i \times c_j$ matrix W_{ij} defined by

$$
W_{ij} = \begin{cases} A_i E_{\rho_i(j),\gamma_j(i)} B_j & \text{if } M_{ij} = 1,\\ 0 & \text{otherwise.} \end{cases}
$$

Definition (weaving)

M: $m \times n$ $(0,1)$-matrix. $M = J$

$$R_i = \{j \mid 1 \le j \le n,\ M_{ij} = 1\}, \quad r_i = |R_i| = n,$$
$$\rho_i : R_i \to \{1, \dots, r_i\} \text{ bijection, } \textit{identity}$$
$$A_i : r_i \times r_i \text{ matrix, } n \times n$$
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$$B_j : c_j \times c_j \text{ matrix. } m \times m$$

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Proposition

The weaving of $(A_1, \ldots, A_m)$ and $(B_1, \ldots, B_n)$ with respect to J is the same as the generalized tensor product $(A_1, \ldots, A_m) \otimes (B_1, \ldots, B_n)$ after row permutaiton.

The method of weaving

Proposition (Craigen, 1991)

M : $m \times n$ $(0,1)$-matrix,	
r_i :row sum of M	$(1 \leq i \leq m)$,
A_i : $r_i \times r_i$ matrix	$(1 \leq i \leq m)$,
c_j :column sum of M	$(1 \leq j \leq n)$,
B_j : $c_j \times c_j$ matrix	$(1 \leq j \leq n)$

The method of weaving

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$$\begin{aligned}
M &: m \times n \ (0,1)\text{-matrix}, \\
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c_j &: \text{column sum of } M && (1 \le j \le n), \\
B_j &: c_j \times c_j \text{ matrix} && (1 \le j \le n)
\end{aligned}$$

Set $N = \sum_{i=1}^{m} r_i = \sum_{j=1}^{n} c_j$.

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Proposition (Craigen, 1991)

$$\begin{aligned} M &: m \times n \ (0,1)\text{-matrix}, & & \\ r_i &: \text{row sum of } M & & (1 \le i \le m), \\ A_i &: r_i \times r_i \text{ matrix} & & (1 \le i \le m), \\ c_j &: \text{column sum of } M & & (1 \le j \le n), \\ B_j &: c_j \times c_j \text{ matrix} & & (1 \le j \le n) \end{aligned}$$

Set $N = \sum_{i=1}^{m} r_i = \sum_{j=1}^{n} c_j$. The weaving of $(A_1, \ldots, A_m)$ and $(B_1, \ldots, B_n)$ with respect to M is a weighing matrix $W(N, ab)$ provided

$$\begin{aligned} A_i &: W(r_i, a) & & (1 \le i \le m), \\ B_j &: W(c_j, b) & & (1 \le j \le n). \end{aligned}$$

The method of weaving, example

M: 6×13 matrix with row sums

$$\begin{aligned} \text{row sums } & 13, 13, 10, 10, 10, 10, \\ A_i : & W(13, 9),\ W(10, 9), \\ \text{column sums } & 6, 6, 6, 6, 6, 6, 6, 4, 4, 4, 4 \\ B_j : & W(6, 4),\ W(4, 4) \end{aligned}$$

Then the weaving gives $W(66, 36)$.

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- Weaving in general cannot be expressed by generalized tensor product.

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Then the weaving gives $W(66, 36)$.

- Weaving in general cannot be expressed by generalized tensor product.
- Perhaps generalized tensor product is not general enough.

$$M = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} 3 \\ 2 \\ 2 \end{matrix}$$

$$\begin{matrix} 3 & 1 & 1 & 2 \end{matrix}$$

		B_1	B_2	B_3	B_4
		$[3 \times 3]$	$[1 \times 1]$	$[1 \times 1]$	$[2 \times 2]$
A_1	$[3 \times 3]$	$A_1 E_{11} B_1$	$A_1 E_{21} B_2$	O	$A_1 E_{31} B_4$
A_2	$[2 \times 2]$	$A_2 E_{12} B_1$	O	$A_2 E_{21} B_3$	O
A_3	$[2 \times 2]$	$A_3 E_{13} B_1$	O	O	$A_3 E_{22} B_4$

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$$\begin{array}{cccccc} & & B_1 & B_2 & B_3 & B_4 \\ & & [3 \times 3] & [1 \times 1] & [1 \times 1] & [2 \times 2] \\ A_1 & [3 \times 3] & A_1 E_{11} B_1 & A_1 E_{21} B_2 & O & A_1 E_{31} B_4 \\ A_2 & [2 \times 2] & A_2 E_{12} B_1 & O & A_2 E_{21} B_3 & O \\ A_3 & [2 \times 2] & A_3 E_{13} B_1 & O & O & A_3 E_{22} B_4 \end{array}$$

$$A_2 E_{21} B_3 = \begin{bmatrix} * & * \\ * & * \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} * \end{bmatrix} = \begin{bmatrix} * & 0 & * & 0 \\ * & 0 & * & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ * \\ 0 \end{bmatrix}$$

$$= \tilde{A}_2 E_{32} \tilde{B}_3.$$

$$M = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & \color{red}{0} & 1 & \color{red}{0} \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} 3 \\ 2 \\ 2 \end{matrix}$$

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		B_1 $[3 \times 3]$	B_2 $[1 \times 1]$	B_3 $[1 \times 1]$	B_4 $[2 \times 2]$
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$$M = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} 3 \\ 2 \\ 2 \end{matrix}$$

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\end{aligned}$$

Define an $r_i \times n$ matrix $\tilde{A}_i$ and an $m \times c_j$ matrix $\tilde{B}_j$ by

$$(\tilde{A}_i)_{hk} = \begin{cases} (A_i)_{h,\rho_i(k)} & \text{if } k \in R_i, \\ 0 & \text{otherwise,} \end{cases}$$

$$(\tilde{B}_j)_{hk} = \begin{cases} (B_j)_{\gamma_j(h),k} & \text{if } h \in C_j, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition

The weaving of $(A_1, \ldots, A_m)$ and $(B_1, \ldots, B_n)$ with respect to M coincides with the generalized tensor product (of variable-order matrices) $(\tilde{A}_1, \ldots, \tilde{A}_m) \otimes (\tilde{B}_1, \ldots, \tilde{B}_n)$ after appropriate row permutation.

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- We may assume without loss of generality $r_1 \geq \cdots \geq r_m$.
- The "diagonal" matrix Δ_{ij} defined by $(\Delta_{ij})_{hh} = (A_h)_{ij}$ is not a square matrix. It is an $s_i \times m$ matrix, where $s_1 \geq \cdots \geq s_{r_1}$ is the conjugate partition of $r_1 \geq \cdots \geq r_m$

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- For example, $(r_1, r_2, r_3) = (3, 2, 2) \implies (s_1, s_2, s_3) = (3, 3, 1)$.

$$
\begin{array}{cccc}
 & s_1 & s_2 & s_3 \\
r_1 & \square & \square & \square \\
r_2 & \square & \square & \\
r_3 & \square & \square &
\end{array}
$$

Proposition

$$H_i : r_i \times n \text{ matrix}, \; H_i H_i^{(-)^\top} = aI \quad (1 \le i \le m),$$

$$K_j : m \times c_j \text{ matrix}, \; K_j K_j^{(-)^\top} = bI \quad (1 \le j \le n).$$

The generalized tensor product (of variable-order matrices) $T = (H_1, \ldots, H_m) \otimes (K_1, \ldots, K_n)$ satisfies $TT^{(-)^\top} = abI$.

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$$x^{(-)} = \begin{cases} x^{-1} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases} \qquad 0^{-1} = 0$$

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and define $X^{(-)}$ similarly for a matrix X.
Observe $(ab)^{(-)} = a^{(-)} b^{(-)}$ $(\forall a, b \in \mathbb{C})$.

Jones graph

- H: $n \times n$ inverse-orthogonal matrix
- $V = \{(i,j) \mid 1 \leq i,j \leq n,\ i \neq j\}$
- The Jones graph of $\Gamma(H)$ is the graph with vertex set V and
$$(i,j) \sim (i',j') \iff \sum_{h=1}^{n} \frac{H_{ih}H_{i'h}}{H_{jh}H_{j'h}} \neq 0.$$

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Theorem (Hosoya–Suzuki)

H is a (nontrivial) generalized tensor product if and only if $\Gamma(H)$ has a connected component contained in $S \times S$ for some $S \subsetneq \{1,\ldots,n\}$.

Strong Kronecker product

Seberry and Zhang (1991) introduced strong Kronecker product:

$$[M_{ij}] \circ [N_{ij}] = [\sum_k M_{ik} \otimes N_{kj}]$$

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$$\begin{aligned} H \otimes K &= (H \otimes I)(I \otimes K) \\ &= (H \otimes \sum_{h=1}^{m} E_{hh})(\sum_{j=1}^{n} E_{jj} \otimes K) \end{aligned}$$

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$$(H_1, \ldots, H_m) \otimes (K_1, \ldots, K_n) = (\sum_{h=1}^{m} H_h \otimes E_{hh})(\sum_{j=1}^{n} E_{jj} \otimes K_j)$$

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$$\begin{bmatrix} K_1 & & \\ & \ddots & \\ & & K_n \end{bmatrix}$$

$$(H_1, \ldots, H_m) \otimes (K_1, \ldots, K_n) = (\sum_{h=1}^{m} H_h \otimes E_{hh})(\sum_{j=1}^{n} E_{jj} \otimes K_j)$$

$$\begin{bmatrix} \Delta_{11} & \cdots & \Delta_{1n} \\ \vdots & & \vdots \\ \Delta_{n1} & \cdots & \Delta_{nn} \end{bmatrix} \begin{bmatrix} K_1 & & \\ & \ddots & \\ & & K_n \end{bmatrix}$$

where $(\Delta_{ij})_{hh} = (H_h)_{ij}$,

$[M_{ij}] \circ [N_{ij}] = [\sum_k M_{ik} \otimes N_{kj}]$

$$(H_1, \ldots, H_m) \otimes (K_1, \ldots, K_n) = (\sum_{h=1}^{m} H_h \otimes E_{hh})(\sum_{j=1}^{n} E_{jj} \otimes K_j)$$

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where $(\Delta_{ij})_{hh} = (H_h)_{ij}$, is a principal submatrix of

$$\begin{bmatrix} & & \\ & \Delta_{ij} \otimes K_j & \\ & & \end{bmatrix} = \begin{bmatrix} \Delta_{11} & \cdots & \Delta_{1n} \\ \vdots & & \vdots \\ \Delta_{n1} & \cdots & \Delta_{nn} \end{bmatrix} \circ \begin{bmatrix} K_1 & & \\ & \ddots & \\ & & K_n \end{bmatrix}$$

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