A parametric family of complex Hadamard matrices

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Hadamard matrices and generalizations

- A (real) Hadamard matrix of order n is an $n \times n$ matrix H with entries ± 1, satisfying $HH^\top = nI$.
- A complex Hadamard matrix of order n is an $n \times n$ matrix H with entries in $\{\xi \in \mathbb{C} \mid |\xi| = 1\}$, satisfying $HH^* = nI$, where $*$ means the conjugate transpose.

Hadamard matrices and generalizations

$$H = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$$

$$HH^\top = 4I$$

$$H = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}$$

$$HH^* = 3I$$

Existence and classification

Conjecture

For any $n \equiv 0 \pmod 4$, a Hadamard matrix of order n exists.

Known for $n \leq 664$. Classified for $n \leq 32$.

For any n, a complex Hadamard matrix of order n exists.
An example is given by the character table of an abelian group of order n.

Complex Hadamard matrices

Classified up to order 5 (unique by Haagerup 1996).
Open for order ≥ 6.

Definition

Two complex Hadamard matrices H_1, H_2 are equivalent if $H_1 = PH_2Q$ for some monomial matrices P, Q whose nonzero entries are complex numbers with absolute value 1.

If n is not a prime, then there are uncountably many inequivalent complex Hadamard matrices, up to equivalence.

Strongly regular graphs

Goethals and Seidel (1970):
symmetric regular Hadamard matrix $\iff$ certain strongly regular graph:

$$H = I + A_1 - A_2, \quad J = I + A_1 + A_2.$$

A_1 = adjacency matrix

Chan and Godsil (2010):
complex Hadamard matrices $\impliedby$ certain strongly regular graph

$$H = I + w_1 A_1 + w_2 A_2, \quad J = I + A_1 + A_2.$$

A_1 = adjacency matrix

Chan (2011):

$$H = I + w_1 A_1 + w_2 A_2 + w_3 A_3, \quad J = I + A_1 + A_2 + A_3.$$

complex Hadamard matrices $\not\Longleftarrow$ certain distance-regular graphs of diameter 3

Ikuta and Munemasa (2014+):
complex Hadamard matrices $\Longleftarrow$ certain symmetric association scheme of class 3.

A_i are pairwise commutative symmetric disjoint (0, 1)-matrices, such that $\langle I, A_1, A_2, A_3 \rangle$ is closed under multiplication (Bose-Menser algebra).

Bose-Mesner algebra

Let A_1, A_2, A_3 be pairwise commutative symmetric disjoint $(0,1)$-matrices satisfying $I + A_1 + A_2 + A_3 = J$, such that $\mathcal{A} = \langle I, A_1, A_2, A_3 \rangle$ is closed under multiplication (Bose-Menser algebra). Then A_i are simultaneously diagonalizable.

Example

Example: Cubic residues in finite fields (Cyclotomic schemes)

$$
\begin{aligned}
V_0 &= \mathrm{Ker}(A_1 - fI) &&= \mathrm{Ker}(A_2 - fI) &&= \mathrm{Ker}(A_3 - fI)\\
V_1 &= \mathrm{Ker}(A_1 - \theta_1 I) &&= \mathrm{Ker}(A_2 - \theta_3 I) &&= \mathrm{Ker}(A_3 - \theta_2 I)\\
V_2 &= \mathrm{Ker}(A_1 - \theta_2 I) &&= \mathrm{Ker}(A_2 - \theta_1 I) &&= \mathrm{Ker}(A_3 - \theta_3 I)\\
V_3 &= \mathrm{Ker}(A_1 - \theta_3 I) &&= \mathrm{Ker}(A_2 - \theta_2 I) &&= \mathrm{Ker}(A_3 - \theta_1 I)
\end{aligned}
$$

Bose-Mesner algebra

Let A_1, A_2, A_3 be pairwise commutative symmetric disjoint $(0,1)$-matrices satisfying $I + A_1 + A_2 + A_3 = J$, such that $\mathcal{A} = \langle I, A_1, A_2, A_3 \rangle$ is closed under multiplication (Bose-Menser algebra). Then A_i are simultaneously diagonalizable.

Definition

$\mathcal{A}$ is called pseudocyclic if the $\mathbb{R}^n = V_0 \oplus V_1 \oplus V_2 \oplus V_3$: common eigenspace decomposition, such that $\downarrow$

$$
\begin{aligned}
V_0 &= \mathrm{Ker}(A_1 - fI) &&= \mathrm{Ker}(A_2 - fI) &&= \mathrm{Ker}(A_3 - fI) \\
V_1 &= \mathrm{Ker}(A_1 - \theta_1 I) &&= \mathrm{Ker}(A_2 - \theta_3 I) &&= \mathrm{Ker}(A_3 - \theta_2 I) \\
V_2 &= \mathrm{Ker}(A_1 - \theta_2 I) &&= \mathrm{Ker}(A_2 - \theta_1 I) &&= \mathrm{Ker}(A_3 - \theta_3 I) \\
V_3 &= \mathrm{Ker}(A_1 - \theta_3 I) &&= \mathrm{Ker}(A_2 - \theta_2 I) &&= \mathrm{Ker}(A_3 - \theta_1 I)
\end{aligned}
$$

Pseudocyclic Bose-Mesner algebra

Conjecture

Given a pseudocyclic Bose-Mesner algebra $\langle I, A_1, A_2, A_3 \rangle$ of order $n = 3f + 1$ with eigenvalues $f, \theta_1, \theta_2, \theta_3$, TFAE:

(i) there are infinitely many complex Hadamard matrices of the form $I + w_1 A_1 + w_2 A_2 + w_3 A_3$,

(ii) $\theta_1, \theta_2, \theta_3$ are not distinct.

(ii) $\iff$ amorphic. We show (ii) $\Longrightarrow$ (i).

$$\begin{aligned}
V_0 &= \mathrm{Ker}(A_1 - fI) &= \mathrm{Ker}(A_2 - fI) &= \mathrm{Ker}(A_3 - fI)\\
V_1 &= \mathrm{Ker}(A_1 - \theta_1 I) &= \mathrm{Ker}(A_2 - \theta_3 I) &= \mathrm{Ker}(A_3 - \theta_2 I)\\
V_2 &= \mathrm{Ker}(A_1 - \theta_2 I) &= \mathrm{Ker}(A_2 - \theta_1 I) &= \mathrm{Ker}(A_3 - \theta_3 I)\\
V_3 &= \mathrm{Ker}(A_1 - \theta_3 I) &= \mathrm{Ker}(A_2 - \theta_2 I) &= \mathrm{Ker}(A_3 - \theta_1 I)
\end{aligned}$$

Amorphic Bose-Mesner algebra

$$
\begin{aligned}
H &= I + w_1 A_1 + w_2 A_2 + w_3 A_3 : \ \text{order } n = q^2, \\
H^* &= I + \overline{w_1} A_1 + \overline{w_2} A_2 + \overline{w_3} A_3, \\
H^{(-)} &= I + \frac{1}{w_1} A_1 + \frac{1}{w_2} A_2 + \frac{1}{w_3} A_3, \\
e_1 &= w_1 + w_2 + w_3, \ e_2 = w_1 w_2 + w_2 w_3 + w_3 w_1, \\
e_3 &= w_1 w_2 w_3.
\end{aligned}
$$

Proposition

Assume $q \geq 4$.
$HH^{(-)} = nI \iff e_1 = -3/(q-1), \ e_2 = e_1 e_3.$

$e_1 = -3/(q-1)$, $e_2 = e_1 e_3$

$$(x - w_1)(x - w_2)(x - w_3) = x^3 - e_1 x^2 + e_2 x - e_3.$$

$$|w_1| = |w_2| = |w_3| = 1 \iff |e_3| = 1.$$

Cohn 1922 gave a general condition for a polynomial equation to have all of its roots on the unit circle (a simpler one by Lakatos and Losoncz, 2009).

For any e_3 with absolute value 1, the complex numbers w_1, w_2, w_3 defined by the cubic equation above, gives a complex Hadamard matrix

$$I + w_1 A_1 + w_2 A_2 + w_3 A_3,$$

of order $n = q^2$, where A_1, A_2, A_3 are the adjacency matrices in an amorphic pseudocyclic Bose-Mesner algebra.

Thank you