A remark on Turyn's construction of conference matrices

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Goldberg (1966) $C(n+1) \implies C(n^3+1)$

Symmetric conference matrix with core C:

$$W = \begin{bmatrix} 0 & \mathbf{1} \\ \mathbf{1}^\top & C \end{bmatrix} : \ (n+1)\times(n+1) \text{ matrix with entries in } \{0, \pm 1\},$$

$$C = C^\top, \quad C \circ I = 0, \quad CJ = 0, \quad C^2 = nI - J.$$

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Then

$$D = C \otimes C \otimes C - I \otimes J \otimes C - C \otimes I \otimes J - J \otimes C \otimes I$$

$$\implies \tilde{W} = \begin{bmatrix} 0 & \mathbf{1} \\ \mathbf{1}^\top & D \end{bmatrix} : \ \text{symmetric conference matrix.}$$

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Assume C: $n \times n$ matrix with entries in $\{0, \pm 1\}$.

$$C = C^\top, \quad CJ = 0, \quad C \circ I = 0, \quad C^2 = nI - J$$

$D = B_0 + B_1 + B_2 + B_3$, where

$$\begin{aligned} B_0 &= C \otimes C \otimes C \\ B_1 &= -I \otimes J \otimes C \\ B_2 &= -C \otimes I \otimes J \\ B_3 &= -J \otimes C \otimes I \end{aligned}$$

$$D^2 = (B_0 + B_1 + B_2 + B_3)^2 = n^3 I - J.$$

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$$\begin{aligned} D^2 &= (B_0 + B_1 + B_2 + B_3)^2 \\ &= B_0^2 + B_1^2 + B_2^2 + B_3^2 \\ &= nI \otimes nI \otimes nI - J \otimes J \otimes J. \end{aligned}$$

Assume C: $n \times n$ matrix with entries in $\{0, \pm 1\}$.

$$C = C^\top, \quad CJ = 0, \quad C \circ I = 0, \quad C^2 = nI - J$$

$D = B_0 + B_1 + B_2 + B_3$, where $J^2 = nJ$

$$\begin{aligned} B_0^2 &= (C \otimes C \otimes C)^2 \\ B_1^2 &= (-I \otimes J \otimes C)^2 \\ B_2^2 &= (-C \otimes I \otimes J)^2 \\ B_3^2 &= (-J \otimes C \otimes I)^2 \end{aligned}$$

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$D = B_0 + B_1 + B_2 + B_3$, where

$$\begin{aligned} B_0^2 &= (nI - J) \otimes (nI - J) \otimes (nI - J) \\ B_1^2 &= I \otimes nJ \otimes (nI - J) \\ B_2^2 &= (nI - J) \otimes I \otimes nJ \\ B_3^2 &= nJ \otimes (nI - J) \otimes I \end{aligned}$$

$$\begin{aligned} D^2 &= (B_0 + B_1 + B_2 + B_3)^2 \\ &= B_0^2 + B_1^2 + B_2^2 + B_3^2 \\ &= nI \otimes nI \otimes nI - J \otimes J \otimes J. \end{aligned}$$

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$D = B_0 + B_1 + B_2 + B_3$, where

$$\begin{aligned}
B_0^2 &= (nI - J) \otimes (nI - J) \otimes (nI - J) \\
B_1^2 &= nI \otimes J \otimes (nI - J) \\
B_2^2 &= (nI - J) \otimes nI \otimes J \\
B_3^2 &= J \otimes (nI - J) \otimes nI
\end{aligned}$$

$$\begin{aligned}
D^2 &= (B_0 + B_1 + B_2 + B_3)^2 \\
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$D = B_0 + B_1 + B_2 + B_3$, where

$$\begin{aligned} B_0^2 &= (nI - J) \otimes (nI - J) \otimes (nI - J) \\ B_1^2 &= -nI \otimes (-J) \otimes (nI - J) \\ B_2^2 &= -(nI - J) \otimes nI \otimes (-J) \\ B_3^2 &= -(-J) \otimes (nI - J) \otimes nI \end{aligned}$$

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$$B_0^2 = (x_1 + y_1)(x_2 + y_2)(x_3 + y_3)$$
$$B_1^2 = -x_1 y_2 (x_3 + y_3)$$
$$B_2^2 = -(x_1 + y_1) x_2 y_3$$
$$B_3^2 = -y_1 (x_2 + y_2) x_3$$

sum $= x_1 x_2 x_3 + y_1 y_2 y_3$

$$
\begin{aligned}
B_0^2 &= (x_1 + y_1)(x_2 + y_2)(x_3 + y_3) \\
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\begin{aligned}
D^2 &= B_0^2 + B_1^2 + B_2^2 + B_3^2 \\
&= nI \otimes nI \otimes nI + (-J) \otimes (-J) \otimes (-J). \\
&= n^3 I - J. \\
&\implies \exists C(n^3 + 1).
\end{aligned}
$$

$$
\begin{aligned}
B_0^2 &= (x_1 + y_1)(x_2 + y_2)(x_3 + y_3) \\
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Seberry (1969) found analogous construction for $C(n^5 + 1), C(n^7 + 1)$.

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Seberry (1969) found analogous construction for $C(n^5 + 1), C(n^7 + 1)$.
Different method by Belevitch (1950) for $C(n^2 + 1)$.

Turyn (1971)

Theorem

$\exists$ symmetric conference matrix of order $n+1$
$\Longrightarrow$ $\exists$ symmetric conference matrix of order n^k+1
for any odd positive integer k.

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for any odd positive integer k.

Assume

$$C = C^\top, \quad C \circ I = 0, \quad CJ = 0, \quad C^2 = nI - J$$

$$D = C \otimes \cdots \otimes C$$

$$\ldots$$

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$\exists$ symmetric conference matrix of order $n+1$
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$k=3$:

$$D = \begin{array}{c} C\otimes C\otimes C \\ -I\otimes J\otimes C \\ -C\otimes I\otimes J \\ -J\otimes C\otimes I \end{array} \quad \text{satisfies} \quad \begin{array}{l} DJ = D\circ I = 0, \\ D^2 = n^3 I - J. \end{array}$$

$$\implies \tilde{W} = \begin{bmatrix} 0 & \mathbf{1} \\ \mathbf{1}^\top & D \end{bmatrix} : \text{ symmetric conference matrix.}$$

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Assume

$$C = C^\top, \quad C \circ I = 0, \quad CJ = 0, \quad C^2 = nI - J$$

$$D = C \otimes \cdots \otimes C$$
$$- \sum \text{replace some } C \otimes C \text{ with } I \otimes J$$

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$$D = C \otimes \cdots \otimes C$$

$$- \sum_{t=1}^{(k-1)/2} \text{replace } t\ C \otimes C\text{'s with } I \otimes J\text{'s}$$

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Summands are disjoint, orthogonal

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Summands are disjoint, orthogonal $D^2 = n^k I - J$?

$C^2 = nI - J,\ J^2 = nJ$

$$D = C \otimes \cdots \otimes C$$
$$- \sum \text{ replace } t\ C \otimes C\text{'s with } I \otimes J\text{'s}$$

$$D^2 = C^2 \otimes \cdots \otimes C^2$$
$$- \sum (\text{replace } t\ C \otimes C\text{'s with } I \otimes J\text{'s})^2$$

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$$D^2 = C^2 \otimes \cdots \otimes C^2$$
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$$D^2 = (nI - J) \otimes \cdots \otimes (nI - J)$$
$$- \sum \text{replace } t\ (nI - J) \otimes (nI - J)\text{'s with } I \otimes nJ\text{'s}$$

$C^2 = nI - J$, $J^2 = nJ$

$$D^2 = C^2 \otimes \cdots \otimes C^2 - \sum(\text{replace } t\ C \otimes C\text{'s with } I \otimes J\text{'s})^2$$

$$D^2 = (nI - J) \otimes \cdots \otimes (nI - J) - \sum \text{replace } t\ (nI - J) \otimes (nI - J)\text{'s with } {\color{red} I \otimes nJ}\text{'s}$$

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$$D^2 = (nI - J) \otimes \cdots \otimes (nI - J) - \sum_{t=1}^{(k-1)/2} (-1)^t \text{replace } t\ (nI - J) \otimes (nI - J)\text{'s with } nI \otimes (-J)\text{'s}$$

$C^2 = nI - J$, $J^2 = nJ$

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$$D^2 = (nI - J) \otimes \cdots \otimes (nI - J) - \sum_{t=1}^{(k-1)/2} (-1)^t \text{replace } t \ (nI - J) \otimes (nI - J)\text{'s with } nI \otimes (-J)\text{'s}$$

$x_i = nI$, $y_i = -J$

$$(x_1 + y_1) \otimes \cdots \otimes (x_k + y_k)$$

$$- \sum_{t=1}^{(k-1)/2} (-1)^t \text{replace } t \ (x_i + y_i) \otimes (x_{i+1} + y_{i+1})\text{'s with } x_i \otimes y_{i+1}\text{'s}$$

$$= x_1 \cdots x_k + y_1 \cdots y_k$$

$$
(x_1 + y_1) \otimes \cdots \otimes (x_k + y_k)
$$

$$
- \sum_{t=1}^{(k-1)/2} (-1)^t \text{replace } t \ (x_i + y_i) \otimes (x_{i+1} + y_{i+1})\text{'s with } x_i \otimes y_{i+1}\text{'s}
$$

$$
= x_1 \cdots x_k + y_1 \cdots y_k
$$

This is a consequence of the inclusion-exclusion, or more generally, the Möbius inversion.

Inclusion-Exclusion

$f : A \to M$, M: abelian group.
$A_1, \dots, A_k \subset A$.

$$\sum_{a \in \bigcup_{i=1}^{k} A_i} f(a) = -\sum_{t=1}^{k} (-1)^t \sum_{|T|=t} \sum_{a \in \bigcap_{i \in T} A_i} f(a).$$

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$f : A \to M$, M: abelian group.
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If $f = 1 : A \to \mathbb{Z}$, then

$$|\bigcup_{i=1}^{k} A_i| = -\sum_{t=1}^{k} (-1)^t \sum_{|T|=t} |\bigcap_{i \in T} A_i|.$$

A weighing matrix of order n and weight w, denoted $W(n,w)$, is a $(0,\pm 1)$ matrix W satisfying $WW^\top = wI$.

Theorem

Let k be odd.

$$\exists W(n_i+1,w)\ (i=1,\dots,k) \implies \exists W(n_1 n_2 \cdots n_k + 1, w^k).$$

Originally formulated by Craigen (1992).

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Hyperplane partitions and difference systems of sets ☆

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