# Self-orthogonal designs and equitable partitions

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## Theorem

The following are equivalent:

1. $\exists$ a self-orthogonal $3$-$(16,8,3\mu)$ design,
2. $\exists$ an equitable partition of the folded halved $8$-cube with quotient matrix

$$\begin{bmatrix} 4(\mu-1) & 4(8-\mu) \\ 4\mu & 4(7-\mu) \end{bmatrix}.$$

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# A $t$-$(v,k,\lambda)$ design $(X,\mathcal{B})$

e.g., $3$-$(16,8,3\mu)$ design

- $X$ is a finite set, $|X| = v$,
- $\mathcal{B} \subset \binom{X}{k} = \{k$-element subsets of $X\}$,
- $\forall T \in \binom{X}{t}$,

$$\lambda = |\{B \in \mathcal{B} \mid B \supset T\}|.$$

Elements of $X$ are called "points", elements of $\mathcal{B}$ are called "blocks".

# Self-orthogonal designs

A $t$-$(v,k,\lambda)$ design $(X,\mathcal{B})$ is self-orthogonal if

$$|B \cap B'| \equiv 0 \pmod 2 \quad (\forall B, B' \in \mathcal{B}).$$

In particular $k \equiv 0 \pmod 2$.

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Let $M$ be the block-point incidence matrix. Then

$$\text{self-orthogonal} \iff MM^\top = 0 \text{ over } \mathbb{F}_2.$$

We call the row space $C$ of $M$ the (binary) code of the design. Then $C \subset C^\perp$.
(Often $C \subset D = D^\perp \subset C^\perp$).

# Hadamard $3$-designs

If $H$ is a Hadamard matrix of order $8n$, i.e., $H$ is a $8n \times 8n$ matrix with entries in $\{\pm 1\}$ satisfying $HH^\top = 8nI$,

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Indeed, after normalizing $H$ so that its first row is $\mathbf{1}$:

$$H = \begin{bmatrix} \mathbf{1} \\ H_1 \end{bmatrix},$$

an incidence matrix is given by

$$M = \frac{1}{2} \begin{bmatrix} J - H_1 \\ J + H_1 \end{bmatrix}.$$

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$3$-$(16, 8, 3)$ Hadamard design is self-orthogonal. Do there exist $3$-$(16, 8, 3\mu)$ designs for $\mu > 1$? (take union?)

# Existence problem

Given $t, v, k, \lambda$, does there exist a $t$-$(v, k, \lambda)$ design?

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In our work we only consider orthogonality mod $2$. The $5$-$(12, 6, 1)$ design of Witt (1938) is not self-orthogonal.

# Binary codes

A $k$-dimensional subspace of $\mathbb{F}_2^n$ is called an $[n,k]$ code. For an $[n,k]$ code $C$, its minimum weight is

$$\min C = \min\{\mathrm{wt}(x) \mid 0 \neq x \in C\}.$$

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and $C$ is called an $[n,k,d]$ code if $d = \min C$.
A code $C$ is doubly even if

$$\mathrm{wt}(x) \equiv 0 \pmod 4 \quad (\forall x \in C),$$

self-orthogonal if

$$C \subset C^\perp,$$

self-dual if

$$C = C^\perp.$$

# 5-designs from binary self-dual codes

A consequence of the Assmus–Mattson theorem:
Doubly even self-dual $[24m, 12m, 4m+4]$ code
$\to$ 5-$(24m, 4m+4, \lambda)$ design.

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- $m = 1$: Witt design; related designs were characterized by Tonchev (1986)
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For $m \geq 3$, existence is unknown:

- $m = 3$ by Harada-M.-Kitazume (2004),
- $m = 4$ by Harada (2005),
- $m \geq 5$ by de la Cruz and Willems (2012).

Lalaude-Labayle (2001), determined binary self-orthogonal codes of min. wt. $k$ whose min. wt. codewords support:

- $3$-design for $k \leq 10$,
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Motivated by spherical analogue:

- Venkov's theorem on spherical designs supported by an even unimodular lattice
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- Nossek (2014): lattices of min $\leq 7$ with spherical $9$-design, min $\leq 9$ with spherical $11$-design, $\nexists$ lattices of min $\leq 11$ with spherical $13$-design.

# Design theoretic viewpoint

Instead of classifying self-orthogonal codes $C$ of min. wt. $k$ such that

$$\mathcal{B} = \{\mathrm{supp}(x) \mid x \in C,\ \mathrm{wt}(x) = k\}$$

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we hope to classify self-orthogonal designs:

$$\mathcal{B} \subset \{x \in C \mid \mathrm{wt}(x) = k\} \subset C \subset C^{\perp}.$$

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More generally, we assume

$$t \geq \left\lfloor \frac{k}{4} \right\rfloor + 1.$$

Note: $k$ is even.
Mendelsohn equations are "overdetermined" system.

# Mendelsohn equations

Let $(X, \mathcal{B})$ be a $t$-$(v, k, \lambda)$ design, $S \subset X$.

$$n_j = |\{B \in \mathcal{B} \mid j = |B \cap S|\}|.$$

Then

$$\sum_{j \geq 1} \binom{j}{i} n_j = \lambda_i \binom{|S|}{i} \quad (i = 1, \dots, t),$$

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a system of $t$ linear equations in unknowns $n_1, n_2, \dots$ (at most $\min\{k, |S|\}$).
If $S \in C^\perp$, then $n_j = 0$ for $j$ odd.
If $S \in \mathcal{B}$ and $k = \min C^\perp$, then $n_j = 0$ for $j > k/2$, so there are $\lfloor k/4 \rfloor$ unknowns.

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**Lemma**

If $(X, \mathcal{B})$ is a self-orthogonal $3$-$(v, k, \lambda)$ design, and the dual code of its code has minimum weight $4$, then $v = 2k \equiv 0 \pmod 4$.

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Recall $3$-$(8, 4, 1)$ Hadamard design exists.
$\nexists$ self-orthogonal $3$-$(12, 6, \lambda)$ design.

# 3-$(16, 8, \lambda)$ design

$$\lambda \leq \binom{16}{8}\binom{8}{3}\binom{16}{3}^{-1} = 1287$$

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  $\implies \lambda \le 18$.

$\lambda = 3$: Hadamard designs.

$\lambda = 6, 9, 12, 15, 18?$
disjoint union?

## Theorem

The following are equivalent:

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$$\begin{bmatrix} 4(\mu-1) & 4(8-\mu) \\ 4\mu & 4(7-\mu) \end{bmatrix}.$$

3. $\mu \in \{1, 2, 3, 4, 5\}$.

In particular, there is no self-orthogonal $3\text{-}(16, 8, 18)$ design.

# The folded halved $8$-cube

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'halved' $=$ even-weight vectors
'folded' $=$ identify with complement

The folded halved $8$-cube $\Gamma$ has $2^6$ vertices, and it is $28$-regular.

$$SRG(64, 28, 12, 12)$$

# Equitable partition

Let $\Gamma$ be a regular graph.
An equitable partition with quotient matrix $Q$ means: the adjacency matrix $A$ is of the form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad A_{ij}\mathbf{1} = q_{ij}\mathbf{1}, \quad Q = (q_{ij}).$$

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$$q_{21}$$

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$$\begin{array}{cl} 8\mu & 64 = |\frac{1}{2}\overline{H(8,2)}| \\ * & \\ 7\mu & 35 = |\frac{1}{2}J(8,4)| \\ & = |J_2(4,2)| = |PG(3,2)| \end{array}$$

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# Self-orthogonal $3$-design

- $\exists$ 3-$(8, 4, 1)$ Hadamard design
- $\nexists$ 3-$(12, 6, \lambda)$ design
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These satisfy $\lfloor k/4 \rfloor + 1 \le t = 3$.

- $\exists$ 5-$(24,12,48)$ design (Uniqueness by Tonchev, 1986)

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More generally, we assume

$$t = \left\lfloor \frac{k}{4} \right\rfloor + 1, \quad k = \min C,$$

but we allow

$$\mathcal{B} \subsetneq \{x \in C \mid \mathrm{wt}(x) = k\}.$$

## Theorem

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Then

$$\frac{2^{2t-1} t \binom{k/2}{k/2-t} \prod_{j=i}^{t-1}(k-j)}{\sum_{i=1}^{t} i(-2)^{i-1} \binom{2t-i-1}{t-1} \binom{k}{i} \prod_{j=i}^{t-1}(v-j)} \in \mathbb{Z}.$$

Note: Given $k$, there are only finitely many $v$ satisfying the conclusion. Lalaude-Labayle: $k \leq 18$. The only $k > 18$ we found which satisfies the conclusion is $k = 24$, $v = 120$, $t = 7$ (but $\nexists$).

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$\exists$ self-orthogonal $t$-$(v,k,\lambda)$ design $(X,\mathcal{B})$ with code $C$, $d^\perp = \min C^\perp$,

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Then

$$\sum_{i=1}^{t} i(-2)^{i-1} \binom{2t-i-1}{t-1} \binom{d^\perp}{i} \prod_{j=i}^{t-1} \frac{v-j}{k-j} = 0.$$

**Problem** Determine all the solutions of this Diophantine equation in $(d^\perp, k, v)$.

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Thank you for your attention!