Self-Orthogonal Designs and Equitable Partitions

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A t-(v, k, λ) design $(X, \mathcal{B})$

- X is a finite set, $|X| = v$,
- $\mathcal{B} \subset \binom{X}{k} = \{k$-element subsets of $X\}$,
- $\forall T \in \binom{X}{t}$,

$$\lambda = |\{B \in \mathcal{B} \mid B \supset T\}| \quad \text{(constant)}.$$

Elements of X are called "points", those of $\mathcal{B}$ "blocks".
Woolhouse (1844), Kirkman (1847), Steiner (1853).

Example

5-$(24, 8, 1)$ design, uniqueness was shown by Witt (1938), with automorphism group M_{24} of Mathieu (1873).

Existence problem

Given t, v, k, λ, does there exist a t-(v, k, λ) design?

Before Teirlinck (1987), only a few t-designs with $t \geq 5$ were known.

Theorem (Teirlinck)

Nontrivial t-designs exist for all $t \geq 1$, i.e.,

$$\forall t \geq 1, \ \exists v, \exists \lambda \text{ s.t. } \exists t\text{-}(v, t+1, \lambda) \text{ design.}$$

CRC Handbook of Combinatorial Designs, 2nd ed. (2006):
$\exists$ 3-$(16, 7, 5)$ design?

A property of 5-$(24, 8, 1)$ design of Witt

$(X, \mathcal{B})$: 5-$(24, 8, 1)$ design.

$$\forall B, B' \in \mathcal{B}, \ |B \cap B'| \in \{0, 2, 4, 8\}.$$

Definition

A t-(v, k, λ) design $(X, \mathcal{B})$ is self-orthogonal if

$$|B \cap B'| \equiv 0 \pmod 2 \quad (\forall B, B' \in \mathcal{B}).$$

In particular $k \equiv 0 \pmod 2$.

Hadamard 3-designs

If H is a Hadamard matrix of order $8n$, i.e., H is a $8n \times 8n$ matrix with entries in $\{\pm 1\}$ satisfying $HH^\top = 8nI$,
$\implies$ a self-orthogonal 3-$(8n, 4n, 2n-1)$ design.
Indeed, after normalizing H so that its first row is $\mathbf{1}$:

$$H = \begin{bmatrix} \mathbf{1} \\ H_1 \end{bmatrix},$$

an incidence matrix is given by

$$M = \frac{1}{2} \begin{bmatrix} J - H_1 \\ J + H_1 \end{bmatrix}.$$

3-$(8, 4, 1)$ Hadamard design is self-orthogonal.
$\nexists 3$-$(8, 4, \lambda)$ designs for $\lambda > 1$

$(X, \mathcal{B})$: self-orthogonal design

Let

$$M = \begin{array}{cc} & X \\ \mathcal{B} & \left[\begin{cases} 1 & B \ni x \\ 0 & B \not\ni x \end{cases}\right] \end{array}$$

be the $|\mathcal{B}| \times |X|$ block-point incidence matrix. Then

$$\text{self-orthogonal} \iff MM^\top = 0 \text{ over } \mathbb{F}_2.$$

We call the row space C of M the (binary) code of the design. Then $C \subset C^\perp$.

For some Hadamard matrix H of order 16, the code C of the design D obtained from H satisfies $C = C^\perp$.

Search $\sigma \in \mathrm{Aut}(C)$, $\mathcal{B} \cup \sigma(\mathcal{B}) \subset C$, using magma (method 2).

Dual weight 4

The dual code $C^\perp$ of the code C of a t-design has minimum weight at least $t+1$.

Lemma

If $(X,\mathcal{B})$ is a self-orthogonal 3-(v,k,λ) design, and the dual code of its code has minimum weight 4, then $v=2k\equiv 0 \pmod 4$.

Recall 3-$(8,4,1)$ Hadamard design exists.
$\nexists$ self-orthogonal 3-$(12,6,\lambda)$ design.
$\exists$ self-orthogonal 3-$(16,8,\lambda)$ design?
$\lambda\equiv 0 \pmod 3$ is necessary.

3-$(16, 8, \lambda)$ design

$$\lambda \leq \binom{16}{8}\binom{8}{3}\binom{16}{3}^{-1} = 1287$$

if we don't require self-orthogonality.

- Divisibility implies $\lambda \equiv 0 \pmod{3}$.

Write $\lambda = 3\mu$.
$\mu = 1$: Hadamard designs.
Need to choose $|\mathcal{B}| = 10\lambda = 30\mu$ blocks out of

$$\binom{16}{8} = 12870.$$

$(X,\mathcal{B})$: self-orthogonal 3-$(16,8,\lambda)$ design

Let C be the code of $(X,\mathcal{B})$. Then

$$C \subset C^\perp$$

so

$$C \subset \tilde{C} = \tilde{C}^\perp \subset C^\perp.$$

There are only two such codes $\tilde{C}$, $e_8 \oplus e_8$ and d_{16}.

d_{16} is the row space of

$$\begin{bmatrix} 01 & 01 & 01 & 01 & 01 & 01 & 01 & 01 \\ 11 & & & & & & & 11 \\ & 11 & & & & & & 11 \\ & & \ddots & & & & & \vdots \\ & & & & & & 11 & 11 \end{bmatrix}$$

The code d_{16}

$$d_{16} = \text{row sp.} \begin{bmatrix} 01 & 01 & 01 & 01 & 01 & 01 & 01 & 01 \\ 11 & & & & & & & 11 \\ & 11 & & & & & & 11 \\ & & \ddots & & & & & \vdots \\ & & & & & & 11 & 11 \end{bmatrix}$$

d_{16} has $128 + 70$ vectors of weight 8,
$64 + 35$ complementary pairs of vectors of weight 8.
Self-orthogonal 3-$(16, 8, \lambda)$ design $(X, \mathcal{B})$ with $\lambda = 3\mu$ has

$$|\mathcal{B}| = 30\mu \quad (15\mu \text{ pairs}).$$

$\mu = 1$: Hadamard 3-design
$\mu = 2$: method 21

$64 + 35$ pairs of vectors of weight 8 in d_{16}

Define a graph structure on $64 + 35$ pairs:

$$\{B_1, B_1^c\} \sim \{B_2, B_2^c\} \iff |B_1 \cap B_2| \in \{2, 6\}.$$

Then

$$64 = \text{folded halved } 8\text{-cube,}$$
$$\text{valence} = 28$$

$$35 = \text{lines of } P^3(\mathbb{F}_2)$$
$$\text{valence} = 16$$

The folded halved 8-cube, $P^3(\mathbb{F}_2)$

The 8-cube is the graph with vertex set $\{0,1\}^8$, two vertices are adjacent whenever they differ by exactly one coordinate.

'halved' $=$ even-weight vectors
'folded' $=$ identify with complement

The folded halved 8-cube Γ has $2^6 = 64$ vertices, and its valence is 28.

The set of 35 lines of $P^3(\mathbb{F}_2)$ naturally carries the structure of a graph.

$64 + 35$ pairs of vectors of weight 8

Need to choose $|\mathcal{B}|/2 = 15\mu$ pairs out of $64 + 35$.
$(X, \mathcal{B})$ is a self-orthogonal 3-$(16, 8, 3\mu)$ design iff

	valence			
8μ	$4(\mu - 1)$	$64 - 8\mu$	$4(7 - \mu)$	$64 =$ folded halved 8-cube
7μ	6μ	$35 - 7\mu$	$4(4 - \mu)$	$35 =$ lines of $P^3(\mathbb{F}_2)$

Easy to find a subgraph of size 7μ, valence 6μ in $P^3(\mathbb{F}_2)$ for $1 \leq \mu \leq 5$.

The folded halved 8-cube

Need to find a partition into two subgraphs (equitable partition)

$$\begin{cases} \text{size } 8\mu & \text{valence } 4(\mu-1) \\ \text{size } 64-8\mu & \text{valence } 4(7-\mu), \end{cases}$$

for $\mu = 3, 4$.

1. $\mu = 4$: find an equitable partition, both of size 32 and valence 12, using magma (method 3).
2. $\mu = 3$: find an equitable partition,

   $$\begin{cases} \text{size } 24 & \text{valence } 8 \\ \text{size } 40 & \text{valence } 16, \end{cases}$$

   using magma (method 4).

Different methods were employed:

1. use an appropriate subgroup of the automorphism group,
2. zero-one optimization.

Zero-one optimization

Need to find an equitable partition,

$$\begin{cases} \text{size } 24 & \text{valence } 8 \\ \text{size } 40 & \text{valence } 16, \end{cases}$$

Let A

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

be the 64×64 adjacency matrix of the folded halved 8-cube. Then

$$A_{11}\mathbf{1} = 8\mathbf{1}, \qquad A_{22}\mathbf{1} = 16\mathbf{1}, \qquad A\mathbf{1} = 28\mathbf{1},$$

so

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ 0 \end{bmatrix} = \begin{bmatrix} 8\mathbf{1} \\ 12\mathbf{1} \end{bmatrix} = \begin{bmatrix} 12\mathbf{1} \\ 12\mathbf{1} \end{bmatrix} - \begin{bmatrix} 4\mathbf{1} \\ 0 \end{bmatrix}.$$

$$Ax = 12\mathbf{1} - 4x, \text{ i.e., } (A + 4I)x = 12\mathbf{1}.$$

Zero-one optimization to solve $(A+4I)x = 121$

Let A be the 64×64 adjacency matrix of the folded halved 8-cube.
We need to find a $(0,1)$-vector x of weight 24 satisfying

$$(A+4I)x = 12\mathbf{1}.$$

$\rightarrow$ method 4.
Search for "maximal" $(0,1)$-vector satisfying

$$(A+4I)x \leq 12\mathbf{1},$$

to see if x has weight 24.

We found an equitable partition

$$\begin{cases} \text{size } \mathbf{24} & \text{valence } \mathbf{8} \\ \text{size } \mathbf{40} & \text{valence } \mathbf{16}, \end{cases}$$

To summarize

Theorem

The following are equivalent:

1. $\exists$ a self-orthogonal $\mathbf{3}$-$(\mathbf{16}, \mathbf{8}, \mathbf{3}\boldsymbol{\mu})$ design,
2. $\exists$ an equitable partition, of the folded halved 8-cube,

$$\begin{cases} \text{size } \mathbf{8}\boldsymbol{\mu} & \text{valence } \mathbf{4}(\boldsymbol{\mu} - \mathbf{1}) \\ \text{size } \mathbf{64} - \mathbf{8}\boldsymbol{\mu} & \text{valence } \mathbf{4}(\mathbf{7} - \boldsymbol{\mu}), \end{cases},$$

3. $\boldsymbol{\mu} \in \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\}$.

Thank you for your attention!