Covering radii and shadows of binary self-dual codes

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We want to determine the image of the mapping

$$\{C \mid C \subset \mathbb{F}_2^n,\ C = C^\perp\} \to \mathbb{Z}[x,y]$$

defined by $C \mapsto W_C(x,y)$, where

$$W_C(x,y) = \sum_{c\in C} x^{n-\mathrm{wt}(c)} y^{\mathrm{wt}(c)},$$

$$\mathrm{wt}(c) = |\{i \mid c_i \neq 0\}| \quad (c \in \mathbb{F}_2^n).$$

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If we restrict the domain to the set of doubly even codes, i.e.,

$$\mathrm{wt}(c) \equiv 0 \pmod 4 \quad (\forall c \in C),$$

then the image is contained in

$$R = \mathbb{Q}[x^8 + 14x^4y^4 + y^8, W_{\mathsf{Golay}}(x, y)]$$

Determining $W_C(x, y)$ for a given C is computationally difficult ($|C| = 2^{n/2}$).

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$$W_{\mathsf{Golay}}(x, y) = x^{24} + y^{24} + 759(x^{16}y^8 + x^8y^{16}) + 2576x^{12}y^{12}.$$

So

$$\dim R_{(n)} = 1 + \lfloor n/24 \rfloor \quad (\text{if } 8 \mid n).$$

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An extremal weight enumerator is the unique homogeneous polynomial of degree n whose coefficient of x^n is 1, and those of

$$\underbrace{x^{n-4}y^4,\ x^{n-8}y^8, \ldots, x^{n-4\lfloor n/24 \rfloor}y^{4\lfloor n/24 \rfloor}}_{\lfloor n/24 \rfloor}$$

are all zero. For example, $W_{\text{Golay}}(x, y)$.
A code C is called extremal if $W_C(x, y)$ is extremal.
Equivalently, C has minimum weight $4\lfloor n/24 \rfloor + 4$, i.e.,

$$\forall c \in C,\ \mathrm{wt}(c) \neq 4, 8, \ldots, 4\lfloor n/24 \rfloor.$$

Extremal doubly even self-dual codes

$C = C^{\perp} \subset \mathbb{F}_2^n$, $8 \mid n$,
all weights$\equiv 0 \pmod 4$,
minimum weight $4\lfloor n/24 \rfloor + 4$.

k	0	1	2	3	≥ 4	
$n = 24k$	–	1	1	???	$?\cdots$	$\nexists\ k \geq 154$
$n = 24k + 8$	1	5	many?	many?	$\cdots$	$\nexists\ k \geq 159$
$n = 24k + 16$	2	16470	many?	many?	$\cdots$	$\nexists\ k \geq 164$

- $n = 72$ open since Sloane (1973).
- Nonexistence for large n by Zhang (1999).
- Uniqueness for $n = 48$ by Houghten–Lam–Thiel–Parker (2003)
- Classification for $n = 40$ by Betsumiya–Harada–M. (2012)

Covering radius

The covering radius $r(C)$ is defined as

$$r(C) = \max\{ \min\{\mathrm{wt}(u) \mid u \in v + C\} \mid v + C \in \mathbb{F}_2^n / C\}.$$

Computationally difficult.
Delsarte bound for extremal doubly even self-dual codes:

$$r(C) \leq \begin{cases} 4k & \text{if } n = 24k, \\ 4k+2 & \text{if } n = 24k+8, \\ 4k+4 & \text{if } n = 24k+16. \end{cases}$$

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k	0	1	2	3	≥ 4
$n = 24k$ $r(C)$	—	1 $4 = 4$	1 $8 = 8$	? ?	$?\cdots$
$n = 24k + 8$ $r(C)$	1	5 $6 = 6$	many $10? = 10$	many? $12, 13 < 14$	$\cdots$
$n = 24k + 16$ $r(C)$	2	16470 $7, 8 \leq 8$	many? ?	many?	$\cdots$

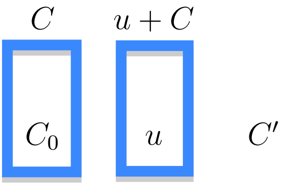
We now focus on the case $n = 24k + 8$.

$n = 24k + 8$

Delsarte: $r(C) \leq 4k + 2$.
Suppose that a coset $u + C$ has minimum weight $4k + 2$. Let

$$C_0 = C \cap \langle u \rangle^{\perp},$$
$$C' = \langle C_0, u \rangle.$$

Then $C' = C'^{\perp}$ has minimum weight $4k + 2$ (not doubly even). $S = C_0^{\perp} \setminus C'$ is called the shadow of C'.



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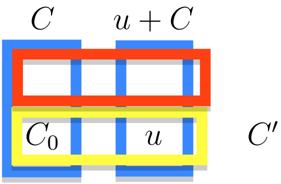

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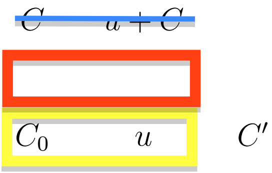

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For $n = 24k + 8$, the following are equivalent:

1. $\exists$ extremal doubly even self-dual code C of length n with covering radius $4k + 2$,
2. $\exists$ self-dual code C' of length n with minimum weight $4k + 2$ and its shadow has minimum weight $4k + 4$.

Bachoc–Gaborit (2004) showed: if a (not doubly even) self-dual code C' of length n with minimum weight d and its shadow has minimum weight s, and

$$2d + s = \frac{n}{2} + 4,$$

then $W_{C'}(x, y)$ and $W_S(x, y)$ are uniquely determined.

$$2(4k + 2) + (4k + 4) = \frac{24k + 8}{2} + 4.$$

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$$W_{C'}(x, y) = \sum_{j=0}^{n/8} a_j (x^2 + y^2)^{n/2-4j} (x^2 y^2 (x^2 - y^2)^2)^j,$$

$$W_S(x, y) = \sum_{j=0}^{n/8} a_j (-1)^j 2^{n/2-6j} (xy)^{n/2-4j} (x^4 - y^4)^{2j},$$

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Thank you for your attention!