On a lower bound on the Laplacian eigenvalues of a graph

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Laplacian Eigenvalue of a Graph

Definition

The Laplacian matrix L of Γ is a matrix indexed by $X = V(\Gamma)$ with

$$L_{xy} = \begin{cases} \deg(x) & \text{if } x = y, \\ -1 & \text{if } x \sim y, \\ 0 & \text{if } x \nsim y. \end{cases}$$

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If $|X| = n$, then the Laplacian eigenvalues are

$$\mu_1 \geq \mu_2 \geq \cdots > \underbrace{0 = \cdots = 0}_{c} = \mu_n,$$

where c is the number of connected components.

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If Γ' is obtained from Γ by deleting an edge, then $\mu_m(\Gamma) \geq \mu_m(\Gamma')$ for all $m \in \{1, 2, \ldots, n\}$.

A Lower Bound

Theorem (Brouwer and Haemers, 2008)

Let Γ have vertex degrees

$$d_1 \geq d_2 \geq \cdots \geq d_n,$$

and Laplacian eigenvalues

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0.$$

Let $m \in \{1, 2, \ldots, n\}$.
If $\Gamma \neq K_m \cup (n-m)K_1$, then $\mu_m \geq d_m - m + 2$.

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- $m = 1$: by Grone and Merris (1994).
- $m = 2$: by Li and Pan (2000).
- $m = 3$: by Guo (2007), and conjectured the above theorem.

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If $\Gamma \neq K_m \cup (n-m)K_1$, then $\mu_m \geq d_m - m + 2$.

$$\mu_m(K_m \cup (n-m)K_1) = 0 = d_m - m + 1.$$

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Problem. Characterize graphs Γ which achieve equality:

$$\mu_m = d_m - m + 2.$$

Proof Technique

Lemma (Interlacing)

Let N be a real symmetric matrix of order n.

$$\lambda_1(N) \geq \cdots \geq \lambda_n(N).$$

If M is a principal submatrix of N, or a quotient matrix of N, with eigenvalues

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then the eigenvalues of M interlace those of N, that is

$$\lambda_i(N) \geq \lambda_i(M) \geq \lambda_{n-m+i}(N) \quad \text{for } i = 1, \ldots, m.$$

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The quotient matrix of N with respect to a partition $X_1, \ldots, X_m$ of $\{1, \ldots, n\}$, have average row sums of N as entries.

Reduction

Assume Γ: graph with n vertices, $1 \le m \le n$.

$$\mu_m = d_m - m + 2.$$

Let $S = \{x_1, \dots, x_m\}$ be a set of vertices with largest degrees:
$\deg x_i = d_i \quad (1 \le i \le m), \quad d_1 \ge \cdots \ge d_m \ge \cdots \ge d_n$.

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provided $\Gamma' \neq K_m \cup (n-m)K_1$.
Unless $S = K_m$ is a connected component, this reduction works, eventually we reach a graph Γ' in which there are no edges outside S.

Γ: graph with n vertices, $1 \leq m \leq n$

Proposition

Assume Γ is edge-minimal subject to d_m.
If Γ satisfies $\mu_m = d_m - m + 2 > \mathbf{0}$, then one of the following holds:

(i) $\mu_m = 1$, *and Γ is K_m with a pending edge attached at a vertex,*

(ii) $\mu_m \geq 2$, *and Γ is K_m with $\mu_m - 1$ pending edges attached at each vertex,*

(iii) $m = 2$ *and* $\Gamma = K_{2,d_m}$.

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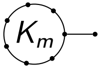

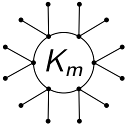

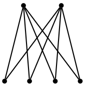

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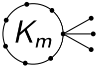

Our contribution for (i): If Δ satisfies $\mu_m = 1 = d_m - m + 2 > 0$, Δ reduces to the case (i) after deleting edges outside S, then Δ is K_m with pending edges attached at the same vertex.

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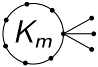

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Similar result is false for (ii) and (iii).

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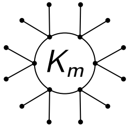
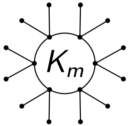


As for the case (ii). . . .

Case (ii) Γ is K_m with $\mu_m - 1$ pending edges at each vertex

$\Delta \longrightarrow \Gamma$



$$L(\Delta) = \begin{bmatrix} (m+\mu_m-1)I - J & -\mathbf{1}^\top \otimes I_m \\ -\mathbf{1} \otimes I_m & {\color{red}M} \end{bmatrix}$$

$$\to \begin{bmatrix} (m+\mu_m-1)I - J & -(\mu_m-1)I_m \\ -I_m & {\color{red}M'} \end{bmatrix} \qquad (2m \times 2m)$$

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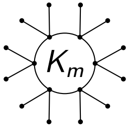
$\Delta \longrightarrow \Gamma$ 

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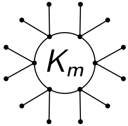
$\Delta \longrightarrow \Gamma$ 

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Δ achieves equality $\implies \lambda_1(M') \leq \frac{m(\mu_m-1)}{m-1}$.

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What if $\mu_m = \mathbf{0}$?

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If $\mu_m = d_m - m + 2 = 0$, then one of the following holds:

(i) $m = 2$ *and* $\Gamma = nK_1$.

(ii) $m = 3$ *and* $\Gamma = 2K_2 \cup (n-4)K_1$.

(iii) $\Gamma = (K_m - tK_2) \cup (n-m)K_1$ *for some* $0 < t \le \lfloor \frac{m}{2} \rfloor$.

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Thank you for your attention!