Group theoretic aspects of the theory of association schemes

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Association schemes

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- P- and Q-polynomial schemes $\to$ Terwilliger algebras
- designs and codes (Delsarte theory) $\to$ SDP
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- P- and Q-polynomial schemes $\rightarrow$ Terwilliger algebras
- designs and codes (Delsarte theory) $\rightarrow$ SDP
- spherical designs $\rightarrow$ Euclidean designs

Permutation representations

association scheme $\approx$ multiplicity-free

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$\subset$ ordinary representation theory

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ordinary representation $\to$ centralizer $=$ algebra
permutation representation $\to$ combinatorial object

Centralizer

Multiplicity-free permutation representation of a finite group G acting on a set X $\rightarrow$ centralizer algebra has a basis consisting of $(0, 1)$-matrices

$$A_0 = I, A_1, \ldots, A_d \text{ with } \sum_{i=0}^{d} A_i = J.$$

These matrices represent orbitals:

$$X \times X = \bigcup_{i=0}^{d} R_i \quad (G\text{-orbits}).$$

Association scheme

Since it is a basis of an algebra,

$$A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k. \qquad (1)$$

$$\{A_0 = I, A_1, \dots, A_d\} = \{A_0^\top = I, A_1^\top, \dots, A_d^\top\}. \qquad (2)$$

This allows one to forget the group G to define an association scheme: $(X, \{R_i\}_{i=0}^d)$; where we require the matrices representing the partition $\{R_i\}_{i=0}^d$ of $X \times X$ satisfy (1) and (2).

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multiplicity-free $\implies$ commutative.

Krein parameters $q_{ij}^k \geq 0$

Multiplicity-free permutation representation of a finite group G acting on a set X has centralizer algebra

$$\mathcal{A} = \langle A_0 = I, A_1, \ldots, A_d \rangle$$

$V = \mathbb{C}^X = L(X)$ decomposes as $\mathbb{C}[G]$-module:

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_d$$

V_0=constant. The orthogonal projections $E_i : V \to V_i$ form another basis of $\mathcal{A}$, so

$$E_i \circ E_j = \frac{1}{|X|} \sum_{i=0}^{d} q_{ij}^k E_k.$$

Scott's theorem

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_d,$$
$$\theta = \chi_0 + \chi_1 + \cdots + \chi_d \quad \text{permutation character}$$

$E_i : V \to V_i$: the orthogonal projection,

$$E_i \circ E_j = \frac{1}{|X|} \sum_{i=0}^{d} q_{ij}^k E_k.$$

Theorem (Scott (1977))

$q_{ij}^k \neq 0 \implies (\chi_i \chi_j, \chi_k) \neq 0.$

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Problem (Bannai and Ito, p.130)

To what extent is the converse of Scott's theorem true?

A counterexample:

$$\mathrm{Ind}_{S_n \times S_n}^{S_{2n}}$$

(Johnson scheme $J(2n, n)$).

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Is there a primitive counterexample?

$q_{ij}^k = 0$ but $(\chi_i\chi_j, \chi_k) \neq 0$?

The primitive counterexample of the smallest degree is

$$G = PGL(2, 11) \text{ acting on } PGL(2, 11)/D_{20}$$

of degree 66.

Eigenvalues of association schemes

Multiplicity-free permutation representation of a finite group G acting on a set $X = G/H$ has centralizer algebra

$$\mathcal{A} = \langle A_0 = I, A_1, \ldots, A_d \rangle$$

$V = \mathbb{C}^X = L(X)$ decomposes as $\mathbb{C}[G]$-module:

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_d$$

V_0=constant. A_i acts on V_j as a scalar:

$$\frac{1}{|H|} \sum_{g \in H a_i H} \chi_j(g).$$

Eigenvalues of association schemes

Multiplicity-free permutation representation of a finite group G acting on a set $X = G/H$ has centralizer algebra

$$\mathcal{A} = \langle A_0 = I, A_1, \ldots, A_d \rangle$$

Eigenvalues of A_i are

$$\{\frac{1}{|H|} \sum_{g \in Ha_iH} \chi_j(g) \mid 0 \le j \le d\} \subset \mathbb{Q}(\exp \frac{2\pi\sqrt{-1}}{e(G)}).$$

where $e(G)$ is the exponent of G.

Splitting fields of association schemes

Question (Bannai and Ito, p.123)

Are there any association schemes in which eigenvalues of A_i's are not all in a cyclotomic number field?

By the Kronecker–Weber theorem, this is equivalent to ask whether the Galois group of the splitting field of the characteristic polynomial of A_i is abelian.

Theorem (M. (1991), Coste–Gannon (1994))

If $q_{ij}^k \in \mathbb{Q}$, then eigenvalues of A_i's are in a cyclotomic number field.

Unlike the converse to Scott's theorem, the question:

Question (Bannai and Ito, p.123)

Are there any association schemes in which eigenvalues of A_i's are not all in a cyclotomic number field?

makes sense for non-commutative association schemes as well.

Splitting fields of association schemes

Let

$$\mathcal{A} = \langle A_0 = I, A_1, \dots, A_d \rangle$$

be an algebra spanned by disjoint $(0,1)$-matrices.

1. $\mathcal{A}$ is the centralizer of a multiplicity-free permutation representation, then the formula for spherical functions implies that eigenvalues of A_i's are cyclotomic.
2. Bannai–Ito asks: the same holds if $\mathcal{A}$ is commutative (without group action)?
3. $\mathcal{A}$ is the centralizer of a non-multiplicity-free permutation representation, then this is not true.

Non-multiplicity-free permutation group

A. Ryba and S. Smith:

$$G = PGL(2,11) \text{ acting on } PGL(2,11)/D_8$$

of degree 165 .

Non-multiplicity-free permutation group

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of degree 165 imprimitive.

$$q_{ij}^k = 0 \text{ but } (\chi_i \chi_j, \chi_k) \neq 0?$$

The primitive example of the smallest degree is

$$G = PGL(2,11) \text{ acting on } PGL(2,11)/D_{20}$$

of degree 66.

Non-cyclotomic numbers

$$\sqrt{3} = e^{\pi\sqrt{-1}/6} + e^{-\pi\sqrt{-1}/6}$$

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If

$$A^2 - cI \quad \text{(distance-2 graph)}$$

has an irrational eigenvalue, then A is likely to have a non-cyclotomic eigenvalue.

Non-multiplicity-free permutation group

A. Ryba and S. Smith:

$$G = PGL(2,11) \text{ acting on } PGL(2,11)/D_{24}$$

has $1+\sqrt{5}$ as an eigenvalue, which is a bipartite half of a bipartite 3-regular graph (flag-transitive incidence structure) on $55+55$ vertices.

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$\implies 4+\sqrt{5}$ is an eigenvalue of A^2
$\implies \sqrt{4+\sqrt{5}}$ is an eigenvalue of A
$\implies 1+\sqrt{4+\sqrt{5}}$ is an eigenvalue of the line graph
(G acts transitively, imprimitively).

Primitive counterexample

1. The smallest primitive group with non-cyclotomic eigenvalue is $PSL(2,19)/D_{20}$.
2. The smallest (imprimitive) transitive group with non-cyclotomic eigenvalue is of degree 32 (due to classification of association schemes by Hanaki).
3. Hanaki and Uno (2006). Even for a prime number of points, the question is unsettled.

Thank you for your attention!